ELSEVIER

Structural Safety 23 (2001) 47–75

STRUCTURAL SAFETY

www.elsevier.nl/locate/strusafe

Moment methods for structural reliability

Yan-Gang Zhao *, Tetsuro Ono

Department of Architecture, Nagoya Institute of Technology, Gokiso-cho, Showa-ku, Nagoya 466-8555, Japan

Abstract

First-order reliability method (FORM) is considered to be one of the most reliable computational methods. In the last decades, researchers have examined the shortcomings of FORM, primarily accuracy and the difficulties involved in searching for the design point by iteration using the derivatives of the performance function. In order to improve upon FORM, several structural reliability methods have been developed based on FORM, such as second-order reliability method (SORM), importance sampling Monte-Carlo simulation, first-order third-moment reliability method (FOTM), and response surface approach (RSA). In the present paper, moment methods for structural reliability are investigated. Five moment method formulas are presented and investigated, and the accuracy and efficiency of these methods are demonstrated using numerical examples. The moment methods, being very simple, have no shortcomings with respect to design points, and requires neither iteration nor the computation of derivatives, and thus are convenient to be applied to structural reliability analysis.

Keywords: FORM; SORM; Reliability index; Failure probability; Performance function; Probability moment; Moment method; Point estimate

1. Introduction

A fundamental problem in structural reliability theory is the computation of the multi-fold probability integral

$$P_{\mathrm{f}} = \mathrm{Prob}[G(\mathrm{X}) \leqslant 0] = \int_{G(\mathrm{X}) \leqslant 0} f(\mathrm{X})\mathrm{d}X \tag{1}$$

where $\mathbf{X} = [X_1, \ldots X_n]^T$, in which the superposed $T=$ transpose, is a vector of random variables representing uncertain structural quantities, $f(\mathbf{X})$ denotes the joint probability density function of $\mathbf{X}$, $G(\mathbf{X})$ is the performance function defined such that $G(\mathbf{X}) \leqslant 0$, the domain of integration, denotes the failure set, and P_{f} is the probability of failure. Difficulty in computing this probability has led to the development of various approximation methods [1], among which the first-order

* Corresponding author. Tel.: +81-52-735-5200; fax: +81-52-735-5570.
E-mail address: yzhao@archi.ace.nitech.ac.jp (Y.-G. Zhao).

PII: S0167-4730(00)00027-8

reliability method (FORM) is considered to be one of the most reliable computational methods [2]. Over the past three decades, contributions from numerous studies have brought FORM to fruition as a reliability method [3–5]. Consequently, FORM has become a basic method for structural reliability. In the last decades, researchers have examined the shortcomings of FORM, primarily accuracy and the difficulties involved in searching for the design point by iteration using the derivatives of the performance function. Several structural reliability methods have been developed based on FORM in an attempt to eliminate these shortcomings. These include the second-order reliability method (SORM) [6–9], importance sampling Monte-Carlo simulation [10,11], first-order third-moment reliability method (FOTM) [12], response surface approach (RSA) [13–15] and genetic algorithm [16,17]. However, the shortcoming associated with the design point or multiple design points [17,18], an inherent drawback in all iteration optimization procedures that are based on mathematical programming methods, appears difficult to overcome.

In the present paper, another route for structural reliability, i.e. moment methods, are investigated. These moment methods, being very simple, have no shortcomings associated with the design point, and require neither iteration nor the computation of derivatives, and thus are convenient to be applied to structural reliability analysis.

2. Review of moment methods

Numerous studies have contributed to the development of reliability methods based on FORM, in which the first- and second-order approximations of the performance function are used and derivative-based iteration search algorithms for the design point are developed. Another route for obtaining structural reliability, i.e. evaluation of the failure probability by directly utilizing the probability moment information of the performance function, has been rarely reported. If the central moments of the performance function can be obtained, the failure probability, which is defined as the probability when the performance function is less than or equal to zero, can be expressed as a function of the central moments. By finding the relationship between the failure probability and the central moments, the failure probability can be obtained. Because the first two moments are generally inadequate, high-order moments will be used.

Approximating the distribution of a random variable using its moments of finite order is a well known problem in statistics and various approximations such as the Pearson, Johnson and Burr systems, the Edgeworth and Cornish-Fisher expansions were developed [19]. For their applications in structural reliability, the Hermite polynomial transformation has been proposed by Winterstein [20], Winterstein and Bjerager [21], and Kohno and Sakamoto [22], and Johnson system has been investigated by Parkinson [23], in which the high order moments were used to approximate the distribution of a basic random variable rather than that of the performance function.

For approximation of the distribution of a performance function using its high order moments, an optimal estimation of convolution integrals using the high order moments of performance function was developed by Grigoriu and Lind [24], in which linear combination of distributions from a prescribed references set were used. Improvement of the accuracy of FORM using high-order moments has been attempted, and a first-order third-moment reliability method (FOTM) [12] and a higher-order moments standardization technique (HOMST) [25] were developed. Both of these methods use the first-order Taylor series of the performance function at the design point, so the results are dependent on the successful search for the design point. For moment methods independent of design point, Grigoriu [26] developed a procedure to estimate the failure probability using Lamda distribution, in which the moments of the performance function were obtained using Monte Carlo simulation. Hong [27] proposed a point-estimate moment-based reliability analysis method, in which the failure probability is approximated by the Johnson family of distributions and the concentrations in the point-estimates are obtained from nonlinear equations. In the present paper, the moments of the performance function are obtained using point estimates in standard normal space in which the concentrations can be easily obtained without solving nonlinear equations. After the moments of the performance function have been obtained, the reliability index of the moment methods can be obtained using some existing standardization function or some existing distribution systems. Five such reliability indices are investigated and the accuracy and efficiency of these methods are demonstrated.

3. Moment method formulas

3.1. Second-moment method

For a performance function $z=G(\mathbf{X})$, if the first two moments are obtained, assuming that $z=G(\mathbf{X})$ obeys normal distribution, the reliability index and failure probability based on the second-moment method are expressed as

$$\beta_{\mathrm{SM}} = \frac{\mu_{\mathrm{G}}}{\sigma_{\mathrm{G}}} \tag{2}$$

$$P_{f\,\mathrm{SM}} = \Phi(-\beta_{\mathrm{SM}}) \tag{3}$$

where and μ_G and σ_G are the mean value and standard deviation of $z=G(\mathbf{X})$, respectively, and Φ is the cumulative distribution function of a normal random variable.

As described above, the computation of μ_G and σ_G should not be based on the first- and second-order approximations of $z=G(\mathbf{X})$, as is the case for the Taylor series.

In addition, as described in the following section, the central moments of the performance function are estimated from the probability distribution of the basic random variables. Therefore, even if only the first two moments of the performance function are used, this method differs from the conventional second-moment method, in which only the first two moments of the basic random variables were used. For a simple performance function, such as the commonly used R-S model in the conventional second-moment method, the first two moments of the performance function can be obtained by using only the first two moments of the basic variables. However, for some complicated performance functions, the first two moments of the performance function may generally not be properly computed by using only the first two moments of the basic variables.

3.2. Third-moment method

For a performance function $z=G(\mathbf{X})$, if the first three moments are obtained, assuming that the standardized variable

$$z_u = \frac{z - \mu_G}{\sigma_G} \tag{4}$$

obeys the three-parameter lognormal distribution (Tichy [12]), the standard normal random variable u is expressed as the following function (see Appendix A):

$$u = \frac{\text{sign}(\alpha_{3G})}{\sqrt{\ln(A)}} \ln\left[\sqrt{A}\left(1 - \frac{z_u}{u_b}\right)\right] \tag{5}$$

where

$$A = 1 + \frac{1}{u_b^2} \tag{6}$$

$$u_b = (a+b)^{\frac{1}{3}} + (a-b)^{\frac{1}{3}} - \frac{1}{\alpha_{3G}} \tag{7}$$

$$a = \frac{1}{\alpha_{3G}}\left(\frac{1}{\alpha_{3G}^2} + \frac{1}{2}\right) \quad b = -\frac{1}{2\alpha_{3G}^2}\sqrt{\alpha_{3G}^2 + 4} \tag{8}$$

α_{3G} is the third dimensionless central moment, i.e. the skewness of $z = G(\mathbf{X})$.

Since

$$\text{Prob}[z \leqslant 0] = \text{Prob}\left[z_u \leqslant -\frac{\mu_G}{\sigma_G}\right] = \text{Prob}[z_u \leqslant -\beta_{SM}] \tag{9}$$

the reliability index and failure probability based on the third-moment method are obtained as

$$\beta_{TM} = \frac{-\text{sign}(\alpha_{3G})}{\sqrt{\ln(A)}} \ln\left[\sqrt{A}\left(1 + \frac{\beta_{SM}}{u_b}\right)\right] \tag{10}$$

$$P_{f\,TM} = \Phi(-\beta_{TM}) \tag{11}$$

Since $u_b > 0$ for $\alpha_{3G} < 0$ and $u_b < 0$ for $\alpha_{3G} > 0$, Eq. (10) is always monotonically increasing for $\beta_{SM} > 0$. Note that when $\alpha_{3G} = 0$, Eq. (10) can not give appropriate results. In this case, the third-moment method is expressed directly as

$$\beta_{TM} = \beta_{SM} \quad \text{for} \quad \alpha_{3G} = 0 \tag{12}$$

3.3. Fourth moment method

For a performance function $z = G(\mathbf{X})$, if the first four moments are obtained, using the principle of HOMST [25], the standardized variable in Eq. (4) is related to a standard normal random variable through the following equation (see Appendix B):

$$u = \frac{\alpha_{3G} + 3(\alpha_{4G} - 1)z_u - \alpha_{3G} z_u^2}{\sqrt{\left(5\alpha_{3G}^2 - 9\alpha_{4G} + 9\right)(1 - \alpha_{4G})}} \tag{13}$$

where α_{4G} is the fourth dimensionless central moment, i.e. the kurtosis of $z = G(\mathbf{X})$.

Using the relationship described in Eq. (9), the reliability index and failure probability based on the fourth-moment method are obtained as

$$\beta_{FM} = \frac{3(\alpha_{4G} - 1)\beta_{SM} + \alpha_{3G}\left(\beta_{SM}^2 - 1\right)}{\sqrt{\left(9\alpha_{4G} - 5\alpha_{3G}^2 - 9\right)(\alpha_{4G} - 1)}} \tag{14}$$

$$P_{fFM} = \Phi(-\beta_{FM}) \tag{15}$$

When $\alpha_{3G} = 0$, Eq. (14) degenerates as $\beta_{FM} = \beta_{SM}$.

One can see that Eq. (14) is monotonically increasing for $\beta_{SM} > 0$. Hereafter, Eq. (14) is denoted as the FM-1 reliability index.

The fourth-moment method can also be applied using the Edgeworth expansion or Gram–Charlier series [19]. The Gram–Charlier series has been investigated by Hong [27] and it is found that the Gram–Charlier series provides unsuitable results. Using the Edgeworth expansion, the probability distribution function of the standardized variable in Eq. (4) is expressed by the first four moments using the following expansion:

$$F(z_u) = \Phi(z_u) - \phi(z_u)\left[\frac{1}{6}\alpha_{3G} H_2(z_u) + \frac{1}{24}(\alpha_{4G} - 3)H_3(z_u) + \frac{1}{72}\alpha_{3G}^2 H_5(z_u)\right] \tag{16}$$

where

$$H_2(x) = x^2 - 1 \tag{17a}$$

$$H_3(x) = x^3 - 3x \tag{17b}$$

$$H_5(x) = x^5 - 10x^3 + 15x \tag{17c}$$

are the second-, third- and fifth-order Hermite polynomials, respectively.

Using the relationship described in Eq. (9), the reliability index and failure probability based on the fourth-moment method are obtained as

$$P_{fFM} = \Phi(-\beta_{SM}) - \phi(\beta_{SM})\left[\frac{1}{6}\alpha_{3G} H_2(-\beta_{SM}) + \frac{1}{24}(\alpha_{4G} - 3)H_3(-\beta_{SM}) + \frac{1}{72}\alpha_{3G}^2 H_5(-\beta_{SM})\right] \tag{18}$$

$$\beta_{FM} = -\Phi^{-1}(P_{fFM}) \tag{19}$$

When $\alpha_{3G}=0$, Eq. (19) degenerates as $\beta_{FM}=\beta_{SM}$.

From the investigated examples in Section 5, one can see that Eq. (18) is generally monotonically increasing. Hereafter, the reliability index expressed by Eqs. (18) and (19) is denoted as the FM-2 reliability index.

Another fourth-moment method is utilizing existing systems of frequency curves, such as the Pearson, Johnson and Burr systems [19], and Ramberg's Lamda distribution [26,28]. The Johnson system has been investigated by Parkinson [23] and Hong [27], and the Lamda distribution has been investigated by Grigoriu [26]. Since the quality of approximating the tail area of a distribution is relatively insensitive to the family selected [19,29] and it is required to solve nonlinear equations in order to determine the parameters of the Johnson and Burr systems [19,30], and the lamda distribution [28], the Pearson system is applied in the present study. For the standardized variable z_u described in Eq. (4), f, the PDF of z_u, satisfies the following differential equation in the Pearson system (see Appendix C).

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}z_u} = -\frac{az_u + b}{c + bz_u + dz_u^2} \tag{20}$$

where

$$a = 10\alpha_4 - 12\alpha_3^2 - 18 \tag{21a}$$

$$b = \alpha_3(\alpha_4 + 3) \tag{21b}$$

$$c = 4\alpha_4 - 3\alpha_3^2 \tag{21c}$$

$$d = 2\alpha_4 = 3\alpha_3^2 - 6 \tag{21d}$$

Solving Eq. (20), f, the PDF of z_u, is listed in Table 1 with different relative values of the parameter a, b, c, d (see Appendix C). Using the relationship described in Eq. (9), the reliability index based on the fourth-moment method is given as:

$$\beta_{FM} = -\Phi^{-1}\left[\int_{-\infty}^{-\beta_{SM}} f(z_u)\mathrm{d}z_u\right] \tag{22}$$

Hereafter, the reliability index expressed by Eq. (22) is denoted as the FM-3 reliability index.

4. Estimation for moments of the performance function

In the present paper, point estimates [31] are used in order to estimate the moments of the performance function. For a function of only one random variable $y=y(x)$, the moments of y can be point-estimated as [31],

Table 1
Probability density functions for the Pearson system[a]

Parameter			PDF $f(z_u)$	Range of z_u	Type
$b\neq 0$	$\Delta > 0$	$d < 0$	$K(z_u - r_2)^{\frac{-1}{\sqrt{\Delta}}(ar_2+b)}(r_1 - z_u)^{\frac{1}{\sqrt{\Delta}}(ar_1+b)}$	(r_2, r_1)	I
		$d = 0$	$K(c + bz_u)^{(ac-b^2)/b^2}\exp\left[-\frac{az_u}{b}\right]$	$\left(-\infty, -\frac{c}{b}\right)$ for $b < 0$ $\left(-\frac{c}{b}, +\infty\right)$ for $b > 0$	III
		$d > 0$	$K\lvert z_u - r_1\rvert^{\frac{1}{\sqrt{\Delta}}(ar_1+b)}\lvert z_u - r_2\rvert^{\frac{-1}{\sqrt{\Delta}}(ar_2+b)}$	$(-\infty, r_1)$ for $b < 0$ $(r_2, +\infty)$ for $b > 0$	VI
	$\Delta = 0$		$K\lvert z_u - r_0\rvert^{-\frac{a}{d}}\exp\left[\frac{ar_0 + b}{d(z_u - r_0)}\right]$	$(-\infty, r_0)$ for $b < 0$ $(r_0, +\infty)$ for $b > 0$	V
	$\Delta < 0$		$K\left(c + bz_u + dz_u^2\right)^{\frac{-a}{2d}}\exp\left[\frac{ab - 2bd}{d\sqrt{-\Delta}}\tan^{-1}\left(\frac{b + 2dz_u}{\sqrt{-\Delta}}\right)\right]$	$(-\infty, +\infty)$	IV
b = 0	$\Delta > 0 (d < 0)$		$K\left(-c/d - z_u^2\right)^{\frac{-a}{2d}}$	$\left(\frac{\sqrt{\Delta}}{2d}, \frac{-\sqrt{\Delta}}{2d}\right)$	II
	$\Delta = 0 (d = 0)$		$\exp\left[-\frac{z_u^2}{2}\right]$	$(-\infty, +\infty)$	N
	$\Delta < 0 (d > 0)$		$K\left(c + dz_u^2\right)^{\frac{-a}{2d}}$	$(-\infty, +\infty)$	VII

[a] Note: $\Delta = b^2 - 4cd$, $r_1 = \frac{-b - \sqrt{\Delta}}{2d}$, $r_2 = \frac{-b + \sqrt{\Delta}}{2d}$, $r_0 = \frac{-b}{2d}$, K is determined by $F_x(+\infty) = 1$.

$$\mu_y = \sum_{k=1}^{m} P_k y\left[T^{-1}(u_k)\right] \tag{23}$$

$$\sigma_y^2 = \sum_{k=1}^{m} P_k \left\{y\left[T^{-1}(u_k)\right] - \mu_y\right\}^2 \tag{24}$$

$$\alpha_{ry}\sigma_y^r = \sum_{k=1}^{m} P_k \left\{y\left[T^{-1}(u_k)\right] - \mu_y\right\}^r \tag{25}$$

where μ_y, σ_y, and α_{ry} are the mean value, standard deviation and the rth dimensionless central moment of $y(x)$, T^{-1} is the inverse Rosenblatt transformation. $u_1, u_2, \ldots, u_m$ are the estimating points and $P_1, P_2, \ldots P_m$ are the corresponding weights.

The estimating points u_1 and their corresponding weights P_i can be readily obtained by [31]

$$u_i = \sqrt{2x_i} \qquad P_i = \frac{w_i}{\sqrt{\pi}} \tag{26}$$

where x_i and w_i are the abscissas and weights for Hermite integration with weight function $\exp(-x^2)$ [32].

For a five-point estimate in standard normal space,

$$u_0 = 0, \qquad P_0 = 8/15 \tag{27a}$$

$$u_{1+} = -u_{1-} = 1.3556262 \qquad P_1 = 0.2220759 \tag{27b}$$

$$u_{2+} = -u_{2-} = 2.8569700 \qquad P_2 = 1.12574 \times 10^{-2} \tag{27c}$$

For a seven-point estimate in standard normal space,

$$u_0 = 0 \qquad P_0 = 16/35 \tag{28a}$$

$$u_{1+} = -u_{1-} = 1.1544054 \qquad P_1 = 0.2401233 \tag{28b}$$

$$u_{2+} = -u_{2-} = 2.3667594 \qquad P_2 = 3.07571 \times 10^{-2} \tag{28c}$$

$$u_{3+} = -u_{3-} = 3.7504397 \qquad P_3 = 5.48269 \times 10^{-4} \tag{28d}$$

For a function of many variables $z = G(\mathbf{X})$, where $\mathbf{X} = x_1, x_2, \ldots, x_n$. The joint probability density is assumed to be concentrated at points in the k^n hyperquadrants of the space defined by the n random variables, where k is the number of estimating points used in the point estimates for functions of single random variables. The computation becomes excessive when n is large.

In the present paper the function $G(\mathbf{X})$ is approximated by the following function:

$$G'(\mathrm{X}) = \sum_{i=1}^{n} \left(G_i - G_\mu\right) + G_\mu \tag{29}$$

where

$$G_\mu = G(\boldsymbol{\mu}) \tag{30}$$

$$G_i = G\left[T^{-1}(U_i)\right] \tag{31}$$

$\boldsymbol{\mu}$ represents the vector in which all the random variables take their mean values, and $\boldsymbol{U}_i = [u_{\mu 1}, u_{\mu 2}, \ldots, u_i, u_{\mu i+1}, \ldots, u_{\mu m}]^T$ where $u_{\mu k}, k = 1, \ldots, n$ except i, is the kth value of u_μ, which is the vector in u-space corresponding to $\boldsymbol{\mu}$. G_μ is a constant and G_i, is a function of only u_i, for a specific $G(\mathbf{X})$. T^{-1} is the inverse Rosenblatt transformation.

Since u_i, $i = 1, \ldots, n$ are dependent and G_i is a function of only u_i, G_i, $i = 1, \ldots, n$ are independent. Therefore, the first four central moments of $G'(\mathbf{X})$ are expressed as [31]

$$\mu_G = \sum_{i=1}^{n} \left(\mu_i - G_\mu\right) + G_\mu \tag{32}$$

$$\sigma_G^2 = \sum_{i=1}^{n} \sigma_i^2 \tag{33}$$

$$\alpha_{3G}\sigma_G^3 = \sum_{i=1}^{n} \alpha_{3i}\sigma_i^3 \tag{34}$$

$$\alpha_{4G}\sigma_G^4 = \sum_{i=1}^{n}\alpha_{4i}\sigma_i^4 + 6\sum_{i=1}^{n-1}\sum_{j>i}^{n}\sigma_i^2\sigma_j^2 \tag{35}$$

where $\mu_i, \sigma_i, \alpha_{3i}$, and α_{4i} are the first four moments of G_i.

Since G_i, is a function of only one standard normal random variable u_i, Eq. (31) can be rewritten as

$$G_i = g_i(u) \tag{36}$$

and $\mu_i, \sigma_i, \alpha_{3i}$, and α_{4i} can be point-estimated from Eqs. (23)–(25).

Note, that the general expressions for the functions $G[T^{-1}(\mathbf{U})]$ in Eq. (25) are not necessary. The Rosenblatt transformation is only required at the estimating points, which is similar as the manner in FORM.

For a performance function $G(\mathbf{X})$, the moment methods presented in this paper can be summarized as the following four steps:

(1) Determine the constant G_μ and the function G_i of only one variable u_i using Eqs. (30) and (31).
(2) Compute the first four moments of G_i using Eqs. (23)–(25), with the estimating points and corresponding weights listed in Eqs. (27) and (28).
(3) Compute the first four moments of the performance function using Eqs. (32)–(35).
(4) Compute the reliability indices presented in the previous section.

For a performance function $G(\mathbf{X})$ with n variables, if the probability moments of G_i are estimated using m-point estimate, only mn function calls of $G(\mathbf{X})$ are required for estimating the first four moments of $G(\mathbf{X})$, and after the first four moments of $G(\mathbf{X})$ have been obtained, the reliability analysis become a problem of approximating the distribution of a specific random variable with its known first four moments. If one uses reliability methods based on FORM, kn (k is the number of iterations) functional calls of $G(\mathbf{X})$ and the gradient of $G(\mathbf{X})$ are required for a common procedure of FORM, and efforts are required in order to improve the shortcomings of FORM, associated with gradients, design points and accuracy, etc.

From the procedure of the point estimates described above, one can see that the moments of the performance function are estimated using the full distribution of random variables rather than the moments of them. Therefore the moment methods described in the present paper belong to full distribution methods.

5. Examples and investigations

5.1. A simple problem of system reliability

The performance function in the first example is defined as the minimum value of 8 linear performance functions listed in Table 2 [33]. Assume x_1, x_2 are independent standard normal random variables.

Table 2
Performance function of Example 1

Performance function	Design point	β
$g_1 = 1.4x_1 - 2x_2 + 7.2$	−1.691, 2.416	2.949
$g_2 = 2x_1 - 2x_2 + 8$	−2.000, 2.000	2.828
$g_3 = 2.6x_1 - 2x_2 + 9.3$	−2.247, 1.729	2.835
$g_4 = -1.5x_1 - 2x_2 + 10$	2.400, 3.200	4.000
$g_5 = 4x_1 - 2x_2 + 14$	−2.800, 1.400	3.130
$g_6 = 0.7x_1 - 2x_2 + 6.8$	−1.060, 3.029	3.209
$g_7 = -0.5x_1 - 2x_2 + 8$	0.941, 3.765	3.881
$g_8 = -2x_1 - 2x_2 + 11$	2.750, 2.750	3.889

$$G = \min\{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\} \tag{37}$$

The example is a series system problem. Since the gradients of the performance function are not convenient to obtain, the example is not convenient to solve directly using the conventional procedure of FORM. The FORM results would be obtained by first locating all the design points and then applying one of the methods to compute the probability of unions. If Ditlevsen's bounds [34] are used, the system reliability index is $2.738 \leqslant \beta \leqslant 2.783$. If the proposed method is used, the reliability analysis can be easily conducted without shortcomings associated with the design points, and do not require iteration or computation of derivatives. G_μ and G_i in Eq. (29) are readily obtained as

$$G_\mu = 6.8 \tag{38a}$$

$$\begin{aligned} G_1 = \min\{&1.4x_1 + 7.2, \quad 2x_1 + 8, \quad 2.6x_1 + 9.3, \quad -1.5x_1 + 10, \quad 4x_1 + 14, \\ &0.7x_1 + 6.8, -0.5x_1 + 8, \quad -2x_1 + 11\} \end{aligned} \tag{38b}$$

$$G_2 = -2x_2 + 6.8 \tag{38c}$$

Using the seven-point estimates described in the previous section, the first four moments of G_1 and G_2 are easily obtained as $\mu_1 = 6.5288$, $\sigma_1 = 0.9087$, $\alpha_{31} = -1.9349$, $\alpha_{41} = 9.0180$, $\mu_2 = 6.8$, $\sigma_2 = 2.0$, $\alpha_{32} = 0$, $\alpha_{42} = 3$. Using Eqs. (32)–(35), the first four moments of G are approximately obtained as $\mu_G = 6.5288$, $\sigma_G = -2.19674$, $\alpha_{3G} = -0.1369$, $\alpha_{4G} = 3.1762$.

The reliability indices obtained using the five formulas described in the present paper are listed in Table 3 along with the corresponding results using Monte-Carlo simulation (MCS) with 100,000 samples. Table 3 reveals that the TM, FM-1, FM-2, FM-3 reliability indices are in close agreement with the MCS results.

5.2. Application as second-order reliability indices

The second example considers the following performance function in standardized space, which was originally introduced by Der Kiureghian [9] to investigate the efficiency of SORM.

Table 3
Reliability indices of Example 1

Kind of β	β	P_F
SM	2.972	1.479×10^{-3}
TM	2.811	2.467×10^{-3}
FM-1	2.814	2.442×10^{-3}
FM-2	2.751	2.971×10^{-3}
FM-3	2.752	2.958×10^{-3}
MCS	2.756	2.930×10^{-3}

$$G(\boldsymbol{U}) = \beta_F - u_n + \frac{1}{2}\sum_{j=1}^{n-1} j\ au_j^2 \tag{39}$$

The exact results used in the following investigations are obtained by the IFFT method [35], and the SORM results are obtained using the empirical second-order reliability index [35]. The first four moments of this performance function are obtained using Eqs. (32)–(35) and five-point estimates for each variable.

Fig. 1 shows the relationship between curvature radius and the reliability indices obtained using the four formulas presented in the present paper. In addition, the exact results and the SORM results, in which the first-order reliability index is taken to be 2.0 and the number of

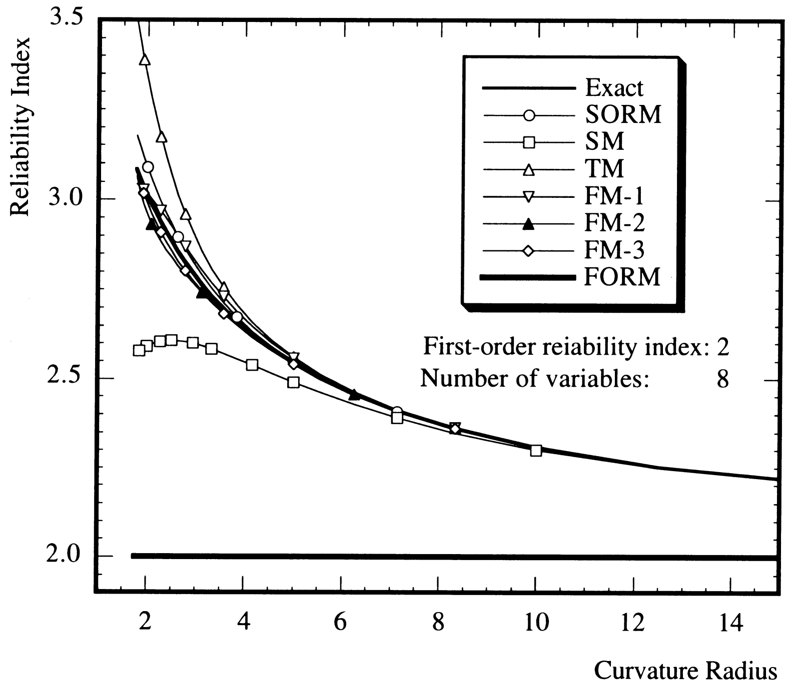

Fig. 1. Variation of reliability index with respect to curvature radius for Example 1.

random variables is taken to be 8, are included for comparison. Fig. 1 shows that all of the reliability indices, including the second-moment reliability index (SM reliability index), presented in the present paper, provide significant improvement of the first-order reliability index, and that all of the reliability indices, including the SM reliability index, give good approximations of the exact results for a relatively large curvature radius (larger than 5) in this case. The SM reliability index and the third-moment reliability index (TM reliability index) have significant errors when the curvature radius is smaller than 5. The FM-1, FM-2 and FM-3 reliability indices all have excellent agreement with the exact results and the SORM results over a large range of curvature radius.

The variations of the reliability indices with respect to the number of random variables n are shown in Fig. 2, in which the first-order reliability index is taken to be 2.0 and the curvature radius is taken to be 5. Fig. 2 reveals that all of the reliability indices, including the SM reliability index, give better results than the first-order reliability index. In addition, all of the reliability indices, except the SM reliability index, give good approximations of the exact results for a relative small n (smaller 15) in this case. However, when n is larger than 15, only the FM-1 and FM-3 reliability indices have excellent agreement with the SORM results and gives a good approximation of the exact results. The TM reliability index generally gives results that are larger than the exact results, whereas the FM-2 reliability index gives results that are generally smaller than the exact results. When n is larger than 35, the FM-2 reliability index gives results worse than even the SM reliability index.

The variations of the reliability indices with respect to the first-order reliability index are shown in Fig. 3, in which the number of random variables is taken to be 8 and the curvature radius is

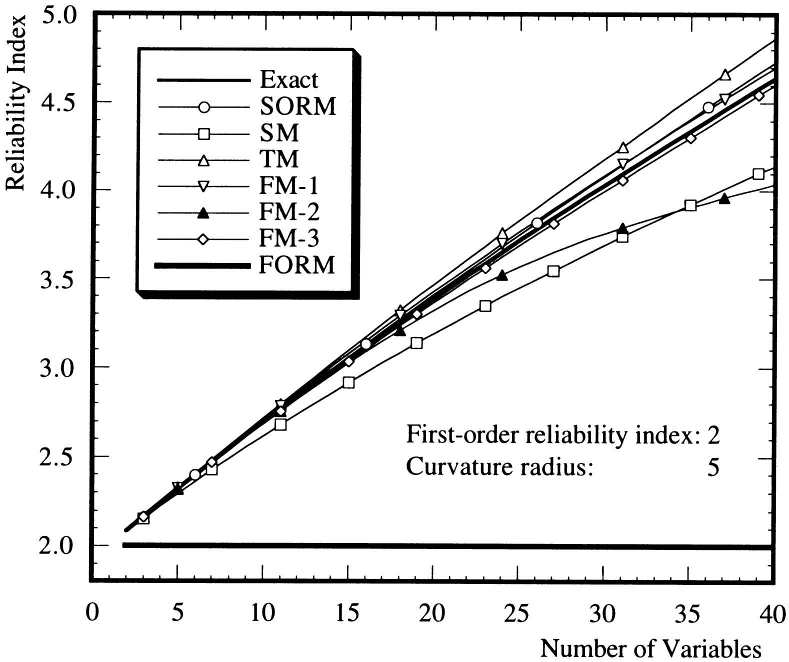

Fig. 2. Variation of reliability index with respect to number of variables for Example 1.

taken to be 5. All of the reliability indices, including the SM reliability index, give better results than the first-order reliability index in this case. In addition, all of the reliability indices, except the SM reliability index, give the good approximations of the exact results for a relative small β_F. However, when β_F is relatively large, only the TM, FM-1 and FM-3 reliability indices give good approximations of the exact results. The FM-2 reliability index gives results worse than even the SM reliability index. When β_F is larger than 4.8, the FM-2 reliability index can not be obtained because probability of failure obtained from Eq. (18) is negative. It is a drawback that the definition of the Edgeworth expansion, Eq. (16), sometimes does not satisfy the definition of the probability distribution function [19].

From this example, the FM-1 reliability index, which is obtained from the HOMST, and the FM-3 reliability index, which is based on the Pearson system of frequency curves, generally give better results than other formulas.

5.3. Parabolic limit state surface with two design points

The third example considers the following parabolic performance function which was proposed by Der Kiuregian and Darkessian [18] and was also investigated by Kuschel et al. [17]:

$$G(\mathbf{U}) = b - u_2 - k(u_1 - e)^2 \tag{40}$$

where $b = 5$, $k = 0.5$ and $e = 0.1$.

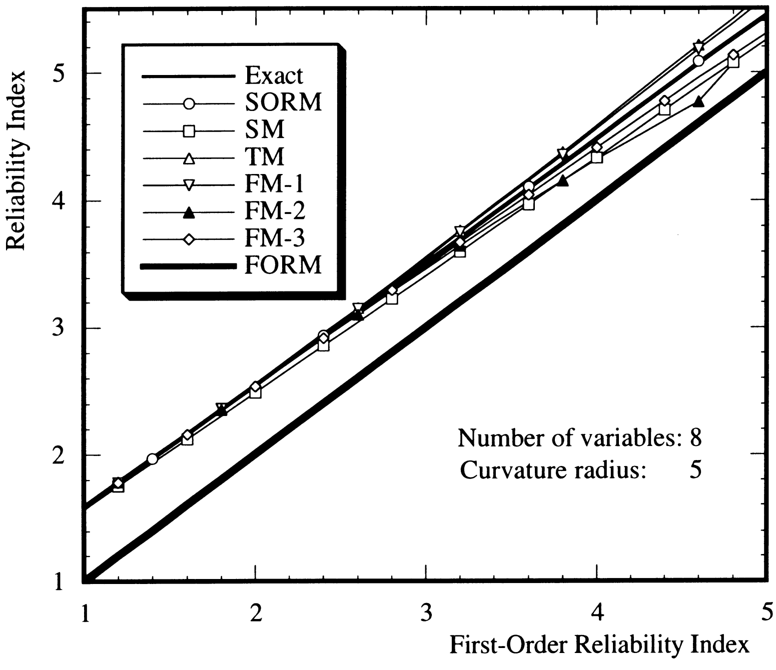

Fig. 3. Variation of reliability index with respect to first-order reliability index for Example 1.

If FORM is used to solve this problem, there are two design points which are successfully obtained by Der Kiuregian and Darkessian [9] as: $\mathbf{U}_1^* = [-2.741,\ 0.965]^T$ with $\beta_1 = 2.906$, $\mathbf{U}_2^* = [2.916,\ 1.036]^T$ with $\beta_2 = 3.094$. If the proposed method is used, by using Eqs. (23)–(35), the first four moments of $G(\mathbf{U})$ can be easily obtained as $\mu_G = 4.4950$, $\sigma_G = 1.2288$, $\alpha_{3G} = -0.5551$, $\alpha_{4G} = 4.3684$, without any necessity to find the multi-design points.

With the aid of the first four moments of $G(\mathbf{U})$, the reliability indices obtained using the 5 formulas presented in the present paper are listed in Table 4 along with the corresponding results using MCS obtained by Der Kiuregian and Darkessian [18]. Table 4 reveals that the TM and FM-2 reliability indices provide comparable results of the two first-order reliability indices, and the FM-3 reliability index is in close agreement with the MCS results. Although the FM-1 reliability index provide comparable results of the two first-order reliability indices, the error in FM-1 reliability index is still significant. This is because when the skewness is quite large and the accuracy of the HOMST is not good [see Appendix B].

5.4. Simple R–S reliability model

In order to investigate the influence of the probability distribution of random variables, the fourth example considers the following performance function, which is an elementary reliability model that is used in many situations,

$$G(\mathbf{X}) = R - S \tag{41}$$

where R is a resistance and S is a load.

Since only two random variables are involved, FORM generally gives good results for this performance function. The first four moments of the performance function can be obtained exactly due to the simplicity of this function. In the following investigations, the coefficient of variance of R is taken to be 0.2 and that of S is taken to be 0.4. The exact results are obtained using MCS for 5,000,000 samplings and the SORM results are obtained using point-fitting SORM [35]. The following six cases are investigated under the assumption that R and S obey different probability distributions.

Table 4
Reliability indices of Example 3

Kind of β	β	P_F
SM	3.658	1.271×10^{-4}
TM	2.914	1.782×10^{-3}
FM-1	3.056	1.120×10^{-3}
FM- 2	2.843	2.224×10^{-3}
FM- 3	2.777	2.741×10^{-3}
MCS	2.751	2.970×10^{-3}
FORM	2.906	1.830×10^{-3}
	3.094	9.900×10^{-4}

Case 1, R is normal and S is lognormal.
Case 2, R is lognormal and S is Weibull.
Case 3, R is normal and S is Weibull.
Case 4, R is lognormal and S is Gumbel (type I extreme value distribution).
Case 5, R is normal and S is Gumbel.
Case 6, R is lognormal and S is Gamma.

For Case 1, the variations of the reliability indices with respect to central factor of safety (CFS) are shown in Fig. 4. From this figure, the TM, FM-1, FM-2 and FM-3 reliability indices are shown to be closer to the MCS results than those of FORM.

For Case 2, the variations of the reliability indices with respect to the CFS are shown in Fig. 5. From this figure, the FM-1 and FM-3 reliability indices show excellent agreement with both the MCS results and the FORM/SORM results while the TM and FM-2 reliability indices have significant errors when the central factor of safety is large.

For Case 3, the variations of the reliability indices with respect to the CFS are shown in Fig. 6. From this figure, the SM, TM, FM-1, FM-2 and FM-3 reliability indices are shown to agree very well with both the MCS and FORM/SORM results.

For Case 4, the variations of the reliability indices with respect to the CFS are shown in Fig. 7. From this figure, only the FM-3 reliability index gives relatively good approximation of the MCS results in the whole investigation range. The FM-1 reliability index gives a good approximation of the MCS results and the FORM/SORM results for small CFS and have significant errors for relatively large CFS due to the reason described previously.

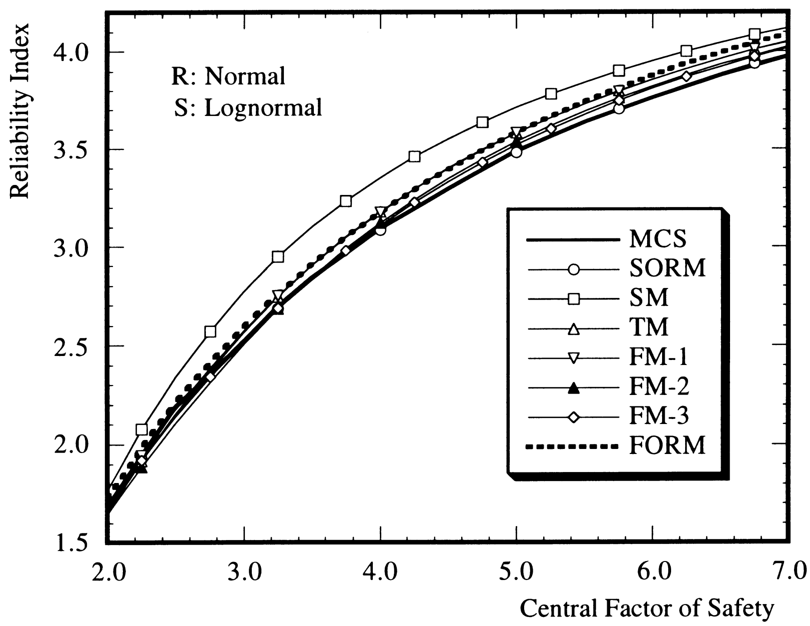

Fig. 4. Relationship between reliability index and central factor of safety for Example 4 (Case 1).

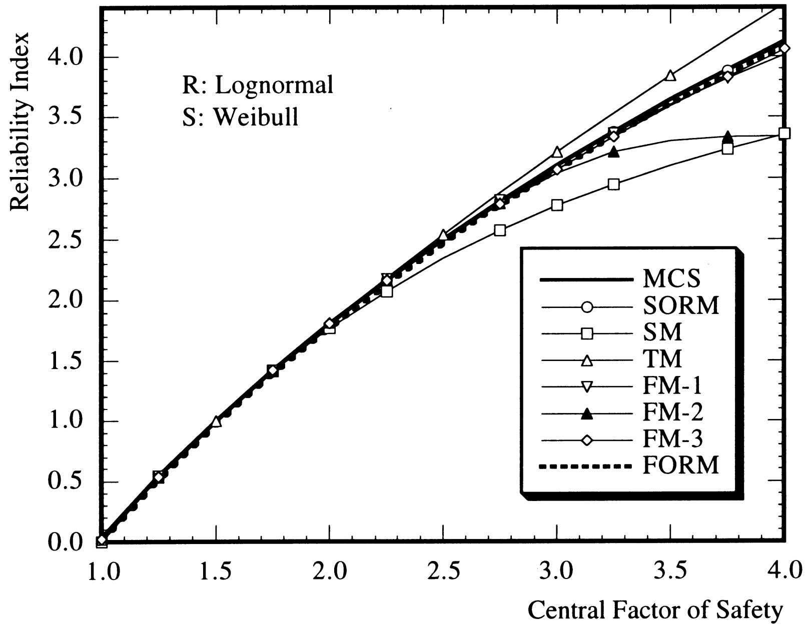

Fig. 5. Relationship between reliability index and central factor of safety for Example 4 (Case 2).

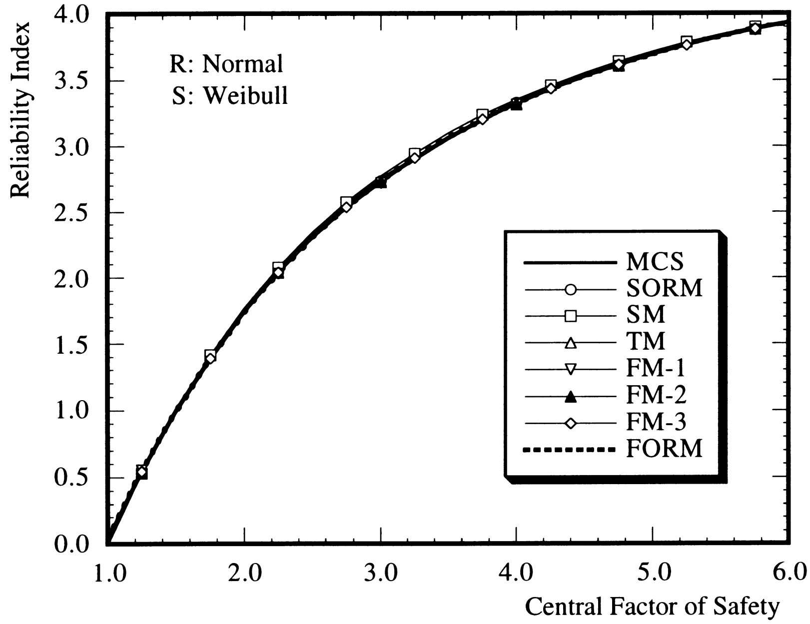

Fig. 6. Relationship between reliability index and central factor of safety for Example 4 (Case 3).

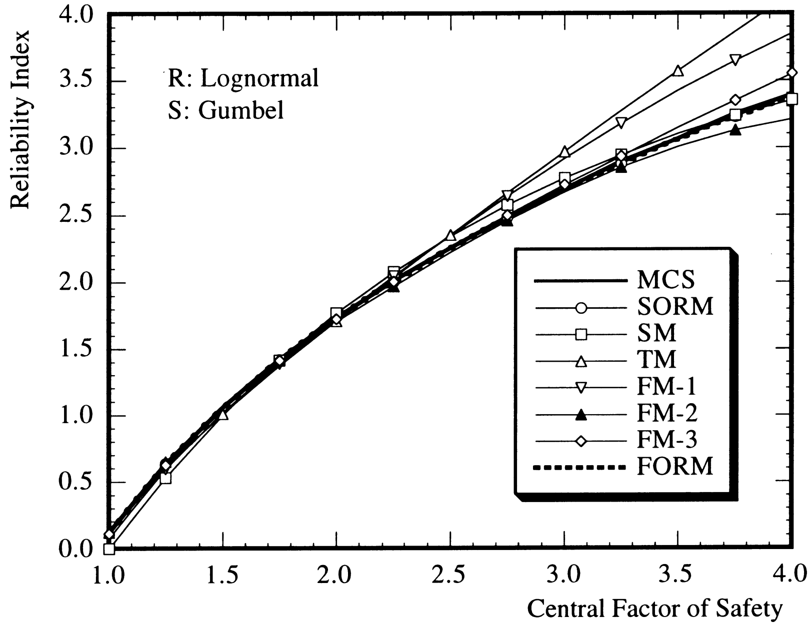

Fig. 7. Relationship between reliability index and central factor of safety for Example 4 (Case 4).

For Case 5, the variations of the reliability indices with respect to the CFS are shown in Fig. 8. From this figure, the TM, FM-1, FM-2 and FM-3 reliability indices are shown to be closer to the MCS results than those of FORM.

For Case 6, the variations of the reliability indices with respect to the CFS are shown in Fig. 9. From this figure, the FM-3 reliability index is shown to agree very well with both the MCS and FORM/SORM results. The FM-1 reliability index gives a good approximation of the MCS results and the FORM/SORM results for small CFS and have significant errors for relatively large CFS because of the reason described previously.

5.5. Example 5

The fifth example considers the following performance function, a plastic collapse mechanism of a one-bay frame, which has been used as example 1 by Der Kiureghian [9], P1217, and Madsen et al. [1], P99.

$$G(\mathbf{X}) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6 \tag{42}$$

The variables x_i are statistically independent and lognormally distributed with the means $\mu_1 = \ldots = \mu_4 = 120$, $\mu_5 = 50$, $\mu_6 = 40$, and standard deviations $\sigma_1 = \ldots = \sigma_4 = 12$, $\sigma_5 = 15$, and $\sigma_6 = 12$.

The FORM reliability index is $\beta_F = 2.348$ corresponding to the failure probability $P_F = 0.00943$. The true value of failure probability is $P_F = 0.0119$.

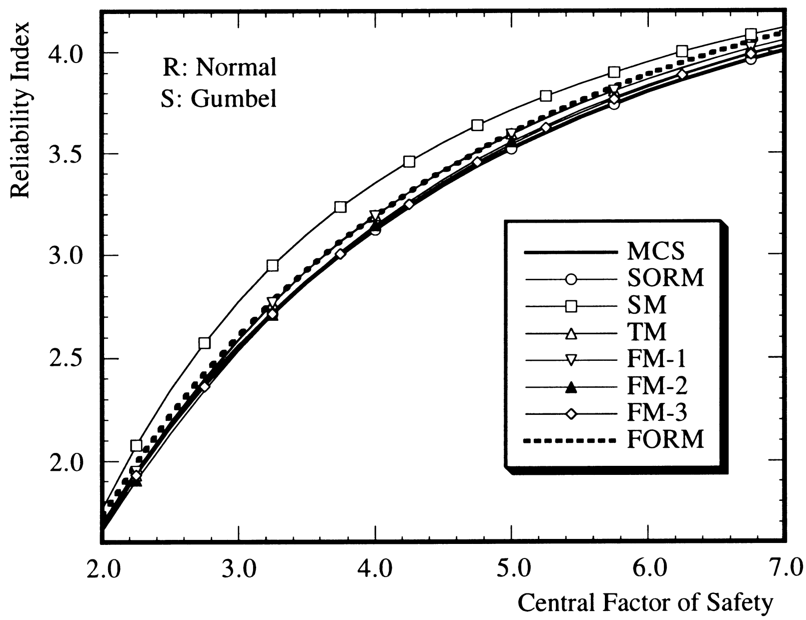

Fig. 8. Relationship between reliability index and central factor of safety for Example 4 (Case 5).

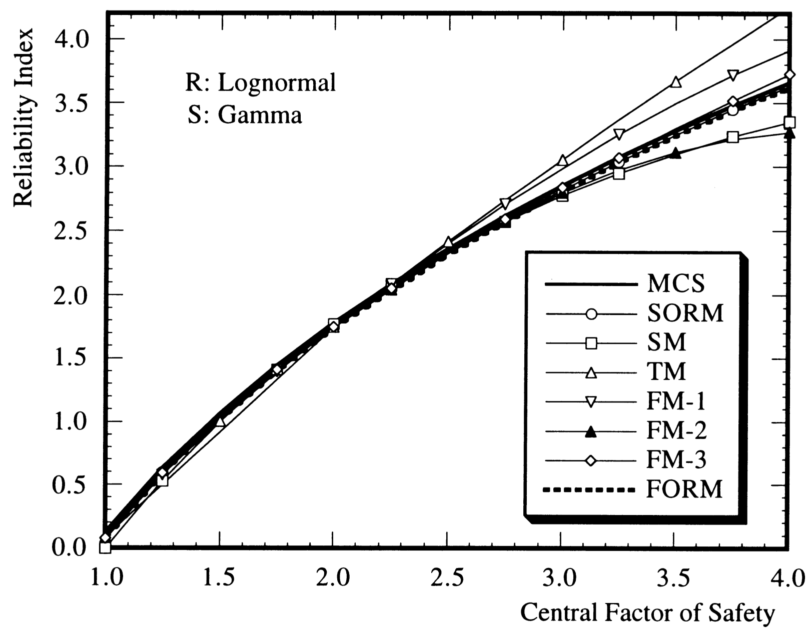

Fig. 9. Relationship between reliability index and central factor of safety for Example 4 (Case 6).

Using Eqs. (32)–(35) and seven-point estimates for each variable, the first four moments of this performance function are obtained as $\mu_G = 270$, $\sigma_G = 103.271$, $\alpha_{3G} = -0.5284$, $\alpha_{4G} = 3.6.5$. With the aid of the first four moments of $G(\mathbf{U})$, the reliability indices obtained using the five formulas presented in the present paper are listed in Table 5 along with the true value. Table 5 reveals that the TM, FM-1, FM-2 and FM-3 reliability indices are in closer agreement with the exact results than FORM.

5.6. Example 6

The sixth example considers the following two simple performance functions.

$$G(\mathbf{X}) = x_1 - x_2 \tag{43a}$$

$$G(\mathbf{X}) = x_1^2 - 2x_2 \tag{44a}$$

In Eq. (43a), the variables x_i are statistically independent and lognormally distributed with the means $\mu_1 = 50$, $\mu_2 = 10$, and standard deviations $\sigma_1 = 10$, $\sigma_2 = 4$. In Eq. (44a), the variables x_i are statistically independent and normally distributed with the means $\mu_1 = 10$, $\mu_2 = 20$, and standard deviations $\sigma_1 = 2$, $\sigma_2 = 5$. Eq. (44a) is suggested by one of the reviewers of this paper.

In order to investigate whether the moment method is sensitive to the formulation of the performance function or not, Eqs. (43a) and (44a) are rewritten as the following equivalent formulation.

$$G(\mathbf{X}) = 1 - \frac{x_2}{x_1} \tag{43b}$$

$$G(\mathbf{X}) = 1 - \frac{2x_2}{x_1^2} \tag{44b}$$

Using Eqs. (32)–(35) and seven-point estimates for each variable, the first four moments of the performance functions Eqs. (43a) and (44a), and their corresponding equivalent formulations Eqs. (43b) and (44b), are listed in Table 6. With the aid of the first four moments, the FM-3 reliability index and its corresponding probability of failure are obtained as shown in Table 6.

Table 5
Reliability indices of Example 5

Kind of β	β	P_F
SM	2.614	4.468×10^{-3}
TM	2.257	1.201×10^{-2}
FM-1	2.290	1.100×10^{-2}
FM-2	2.207	1.367×10^{-2}
FM-3	2.251	1.219×10^{-2}
Exact	2.260	1.190×10^{-2}
FORM	2.348	9.430×10^{-3}

Table 6
Computational results of Example 6

	Performance function Eq. (43)		Performance function Eq. (44)	
	Eq. (43a)	Eq. (43b)	Eq. (44a)	Eq. (44b)
μ_G	40	0.792	64	0.538
σ_G	10.77	0.0902	41.62	0.289
α_{3G}	0.422	−0.942	0.540	−5.951
α_{4G}	3.547	4.869	3.414	102.9
β_{SM}	3.714	8.783	1.538	1.861
β_{FM}	4.262	4.310	1.733	1.704
P_F	1.01×10^{-5}	8.16×10^{-6}	0.0416	0.0442

From Table 6, one can see that although the first four moments and the SM reliability index obtained from Eqs. (43a) and (44a) are much different from those obtained from their corresponding equivalent formulations Eqs. (43b) and (44b), the FM-3 reliability index and its corresponding probability of failure are only slightly different from each other. That is to say, the results of reliability analysis using fourth moment method are not so sensitive to the formulation of the performance functions.

6. Conclusions

(1) Moment methods for structural reliability were investigated. These methods, being very simple, have no shortcomings associated with the design points, and do not require iteration or computation of derivatives, and thus are expected to be convenient to be applied to structural reliability analysis.

(2) Five reliability indices, i.e. the SM, TM, FM-1, FM-2 and FM-3 reliability indices for the moment method were presented and investigated through several numerical examples. The SM reliability index, being very simple, generally has significant errors because the first two moments are generally not adequate for obtaining the failure probability. The TM reliability index, which is obtained from Tichy's FOTM, being more complicated than the FM-1 and FM2 reliability indices, generally produces larger errors than the FM-1 reliability index. The FM-2 reliability index, which is based on the Edgeworth expansion, can not always be obtained due to the fact that the definition of the Edgeworth expansion sometimes does not satisfy the definition of the probability distribution function.

(3) The FM-1 reliability index, which is obtained from the HOMST, generally gives better results than other formulas of comparable simplicity. However, when the skewness of the performance function is large, the FM-1 reliability index produces significant errors.

(4) The FM-3 reliability index, which is obtained from the Pearson system of frequency curves, generally gives suitable results, and is recommended as the reliability index for the moment method.

Compared to the reliability methods based on FORM that have been developed over a period of more than three decades, the moment method for structural reliability has been rarely reported.

However, being very simple, the moment methods are expected to be more convenient and more reliable for engineers if a more simple and accurate reliability index can be developed through further research.

Acknowledgements

The writers would like to thank Prof. Mamoru Kohno, Nagoya University, and Professor Tuyoshi Takada, University of Tokyo, for their beneficial discussions. The writers also wish to thank the reviewers of this paper for their critical comments and suggestions.

Appendix A. Derivation of the third-moment formulas

According to the definition of the three-parameter lognormal probability distribution proposed by Tichy [12], the probability distribution of the standardized random variable for a performance function $z = G(\mathbf{X})$, the standardized random variable z_u defined in Eq. (4) is lognormal when the distribution of

$$u = \ln|z_u - u_b| \tag{A1}$$

is normal. Here, μ_G and σ_G are the mean and standard deviation of z, respectively, and u_b is the standardized bound of the distribution. The inverse distribution function is given by [12]

$$z_u = u_b - \frac{u_b}{\sqrt{A}} \exp\left[\mathrm{sign}(\alpha_{3G}) u \sqrt{\ln(A)}\right] \tag{A2}$$

where

$$A = 1 + \frac{1}{u_b^2} \tag{A3}$$

From Eq. (A2), the following equation can be obtained:

$$u = \frac{\mathrm{sign}(\alpha_{3G})}{\sqrt{\ln(A)}} \ln\left[\sqrt{A}\left(1 - \frac{z_u}{u_b}\right)\right] \tag{A4}$$

and using the relationship of Eq. (9), the reliability index based on the third-moment method is obtained as

$$\beta_{TM} = \frac{-\mathrm{sign}(\alpha_{3G})}{\sqrt{\ln(A)}} \ln\left[\sqrt{A}\left(1 + \frac{\beta_{SM}}{u_b}\right)\right] \tag{A5}$$

The relationship between the bound and the coefficient of skewness is given by

$$\alpha_{3G}u_b^3 + 3u_b^2 + 1 = 0 \tag{A6}$$

The analysis of Eq. (A6) yields

$$u_b = (a+b)^{\frac{1}{3}} + (a-b)^{\frac{1}{3}} - \frac{1}{\alpha_{3G}} \tag{A7}$$

where

$$a = -\frac{1}{\alpha_{3G}}\left(\frac{1}{\alpha_{3G}^2} + \frac{1}{2}\right) \quad b = -\frac{1}{2\alpha_{3G}^2}\sqrt{\alpha_{3G}^2 + 4} \tag{A8}$$

α_{3G} is the coefficient of skewness of $z = G(\mathbf{X})$.

Appendix B. Higher-order moments standardization technique

The high-order moment standardization technique [25] involves a polynomial transformation including determinative coefficients that are determined by setting the first k central moment of the transformed variable equal to those of the standard normal random variable. In a common case, the third-moment standardization function for the standardized random variable z_u defined in Eq. (4) is assumed to be [25]

$$y = z_u + cz_u^2 \tag{B1}$$

$$u = \frac{y - \mu_y}{\sigma_y} \tag{B2}$$

where c is a determinative coefficient.

In order to make Eq. (B1) third moment standardized, the skewness of y should be equal to that of the normal random variable. Then c can be determined by the following equation:

$$\alpha_{3y}\sigma_y^3 = (\alpha_{6z} - 3\alpha_{4z} + 2)c^3 + 3(\alpha_{5z} - 2\alpha_{3z})c^2 + 3(\alpha_{4z} - 1)c + \alpha_{3z} = 0 \tag{B3}$$

where α_{3y} and σ_y are the skewness and standard deviation of y, respectively. α_{3z} α_{4z}, α_{5z} and α_{6z} are the third, fourth, fifth and sixth dimensionless central moments of z_u, they are equal to those of z respectively, according to the definition of probability moments.

According to the computational experience of Ono and Idota [25], c is obtained as shown below, by assuming $|c| << 1$.

$$c = \frac{\alpha_{3z}}{3(1 - \alpha_{4z})} \tag{B4}$$

Since

$$\mu_y = c$$

$$\sigma_y^2 = (\alpha_{4z} - 1)c^2 + 2\alpha_{3z}c + 1$$

the third moment standardization function can be obtained as Eq. (13). Further more, from Eq. (B4), one can see that to ensure the prerequisite $|c| << 1$, α_{3z} should be quite small.

Appendix C. Derivation of Eq. (20) and Table 1

For every member of the Pearson system, $f(y)$, the PDF of a random variable y, satisfies a differential equation of form [19]:

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}y} = -\frac{e + y}{c_0 + c_1 y + c_2 y^2} \tag{C1}$$

in which

$$c_0 = \frac{\mu_2(4\beta_2 - 3\beta_2)}{(10\beta_2 - 12\beta_1 - 18)} \tag{C2a}$$

$$e = c_1 = \frac{\sqrt{\beta_1}\sqrt{\mu_2}(\beta_2 + 3)}{(10\beta_2 - 12\beta_1 - 18)} \tag{C2b}$$

$$c_2 = \frac{(2\beta_2 - 3\beta_1 - 6)}{(10\beta_2 - 12\beta_1 - 18)} \tag{C2c}$$

$\beta_1 = \alpha_3^2$, $\beta_2 = \alpha_4$ are the first and second moment ratios, and μ_2 is the second-order central moment, respectively, of y.

Without loss of generality we write

$$x = \frac{y - \mu}{\sigma} \tag{C3}$$

where μ and σ are the mean value and standard deviation of y, respectively. Then, $\mu_2 = 1$, Eq. (C1) can be rewritten as

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{ax + b}{c + bx + dx^2} \tag{C4}$$

where parameter a, b, c, d are expressed in Eq. (21).

Since for an arbitrary random variable, $\alpha_4 \geqslant \alpha_3^2 + 1$ [19], then $c = 4\alpha_4 - 3\alpha_3^2 \geqslant 4$ always holds true. Various types of the PDF of x can be obtained as following, according to the relative values of b and d.

For $\Delta = b^2 - 4cd > 0$, $b \neq 0$, there are two real roots for equation $c + bx + dx^2 = 0$:

$$r_1 = \frac{-b - \sqrt{\Delta}}{2d}, \qquad r_2 = \frac{-b + \sqrt{\Delta}}{2d}$$

When $d < 0$, since $\Delta > b^2$, the two roots are of the opposite signs and $r_1 > 0$, $r_2 < 0$, for either $b > 0$ or $b < 0$, Eq. (C4) is equivalent to

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{ax + b}{d(r_1 - x)(x - r_2)} \tag{C5}$$

f can be given as

$$f(x) = K(x - r_2)^{\frac{-1}{\sqrt{\Delta}}(ar_2+b)}(r_1 - x)^{\frac{1}{\sqrt{\Delta}}(ar_1+b)} \tag{C6}$$

where K is determined by $F_x(+\infty) = 1$.

Eq. (C6) is corresponding to the Pearson type I distribution. The range of x should be limited so that $r_1 - x > 0$ and $x - r_2 > 0$, which means

$$r_2 < x < r_1$$

When $d = 0$, Eq. (C4) degenerates as

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{ax + b}{c + bx} \tag{C7}$$

f can be given as

$$f(x) = K(c + bx)^{\left(ac-b^2\right)/b^2 \exp\left[-\frac{ax}{b}\right]} \tag{C8}$$

the range of x should be limited so that $c + bx > 0$, which means $x > -b/c$ for $b > 0$ and $x < -b/c$ for $b < 0$. Eq. (C8) is corresponding to the Pearson type III distribution.

When $d > 0$, since $\Delta < b^2$, the two roots are of the same sign and $r_1 < r_2$. If $b > 0$, then, $r_1 < r_2 < 0$, Eq. (C4) is equivalent to

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{ax + b}{d(x - r_1)(x - r_2)} \tag{C9}$$

f can be given as

$$f(x) = K(x - r_1)^{\frac{1}{\sqrt{\Delta}}(ar_1+b)}(x - r_2)^{\frac{-1}{\sqrt{\Delta}}(ar_2+b)} \tag{C10}$$

The range of x should be limited so that $x - r_1 > 0$ and $x - r_2 > 0$, which means

$$x > r_2$$

If $b < 0$, then, $0 < r_1 < r_2$, Eq. (C4) is equivalent to

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{ax+b}{d(r_1 - x)(r_2 - x)} \tag{C11}$$

f can be given as

$$f(x) = K(r_1 - x)^{\frac{1}{\sqrt{\Delta}}(ar_1+b)}(r_2 - x)^{\frac{-1}{\sqrt{\Delta}}(ar_2+b)} \tag{C12}$$

The range of x should be limited so that $r_1 - x > 0$ and $r_2 - x > 0$, which means

$$x < r_1$$

Eq. (C10) and (C12) can be summarized as

$$f(x) = K|x - r_1|^{\frac{1}{\sqrt{\Delta}}(ar_1+b)}|x - r_2|^{\frac{-1}{\sqrt{\Delta}}(ar_2+b)} \quad x < r_1 \text{ for } b < 0, \quad x > r_2 \text{ for } b > 0 \tag{C13}$$

Eq. (C13) is corresponding to the Pearson type VI distribution.

For $\Delta = b^2 - 4cd = 0$, $b \neq 0$ there is only one real root for equation $c + bx + dx^2 = 0$,

$$r_1 = r_2 = r_0 = -\frac{b}{2d}$$

Since $c > 0$, we have $d > 0$. Therefore, $r_0 > 0$ for $b < 0$ and $r_0 < 0$ for $b > 0$.

Eq. (C4) degenerates as

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{ax+b}{(x - r_0)^2} \text{ for } b > 0 \tag{C14}$$

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{ax+b}{(r_0 - x)^2} \text{ for } b < 0 \tag{C15}$$

respectively and f can be given as

$$f(x) = K(x - r_0)^{-\frac{a}{d}}\exp\left[\frac{ar_0 + b}{d(x - r_0)}\right] \quad x > r_0 \tag{C16}$$

$$f(x) = K(r_0 - x)^{-\frac{a}{d}}\exp\left[\frac{ar_0 + b}{d(x - r_0)}\right] \quad x < r_0 \tag{C17}$$

Eqs. (C16) and (C17) can be summarized as

$$f(x) = K|x - r_0|^{-\frac{a}{d}} \exp\left[\frac{ar_0 + b}{d(x - r_0)}\right] \quad x < r_0 \text{ for } b < 0, \;\; x > r_0 \text{ for } b > 0 \tag{C18}$$

Eq. (C18) is corresponding to the Pearson type V distribution.

For $\Delta = b^2 - 4cd < 0$, $b \neq 0$ there is no real root for equation $c + bx + dx^2 = 0$, Eq. (C4) is equivalent to

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{ax + b}{(c - b^2/d/4) + d(x + b/d/2)^2} \tag{C19}$$

f can be given as

$$f(x) = K\left(c + bx + dx^2\right)^{\frac{-a}{2d}} \exp\left[\frac{ab - 2bd}{\mathrm{d}\sqrt{-\Delta}} \tan^{-1}\left(\frac{b + 2dx}{\sqrt{-\Delta}}\right)\right] \tag{C20}$$

since $c + bx + dx^2 > 0$ always hold true in this case, the range of x is $-\infty < x < +\infty$. Eq. (C20) is corresponding to the Pearson type IV distribution.

For $\Delta = b^2 - 4cd > 0$, $b = 0$, which means $d < 0$, $\alpha_3 = 0$, $\alpha_4 < 3$ there are two real roots for equation $c + bx + dx^2 = 0$:

$$r_1 = -\frac{\sqrt{\Delta}}{2d} > 0, \;\; r_2 = \frac{\sqrt{\Delta}}{2d} < 0$$

f can be given as

$$f(x) = K\left(-\frac{c}{d} - x^2\right)^{\frac{-a}{2d}} \tag{C21}$$

The range of x should be limited so that $r_1 - x > 0$ and $x - r_2 > 0$, which means

$$\frac{\sqrt{\Delta}}{2d} < x < -\frac{\sqrt{\Delta}}{2d}$$

Eq. (C21) is corresponding to the Pearson type II distribution.

For $\Delta = b^2 - 4cd = 0$, $b = 0$, which means $d = 0, \alpha_3 = 0, \alpha_4 = 3$, Eq. (C4) degenerates as

$$\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{ax}{c} \tag{C22}$$

f can be given as

$$f(x) = K\exp\left[-\frac{ax^2}{2c}\right] \tag{C23}$$

Eq. (C23) is corresponding to the PDF of a standard normal random variable. Since $a=c=12$, Eq. (C23) can be expressed as

$$f(x)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right) \tag{C24}$$

For $\Delta=b^2-4cd<0$, $b=0$, which means $d>0, \alpha_3=0, \alpha_4>3$, there is no real root for equation $c+bx+dx^2=0$.

f can be given as

$$f(x)=K\left(\frac{c}{d}+x^2\right)^{\frac{-a}{2d}} \quad -\infty<x<\infty \tag{C25}$$

Eq. (C25) is corresponding to the Pearson type VII distribution.

The results described above are summarized in Table 1 and the division of the $\alpha_3^2-\alpha_4$ plane among the various types is exhibited in Fig. 10.

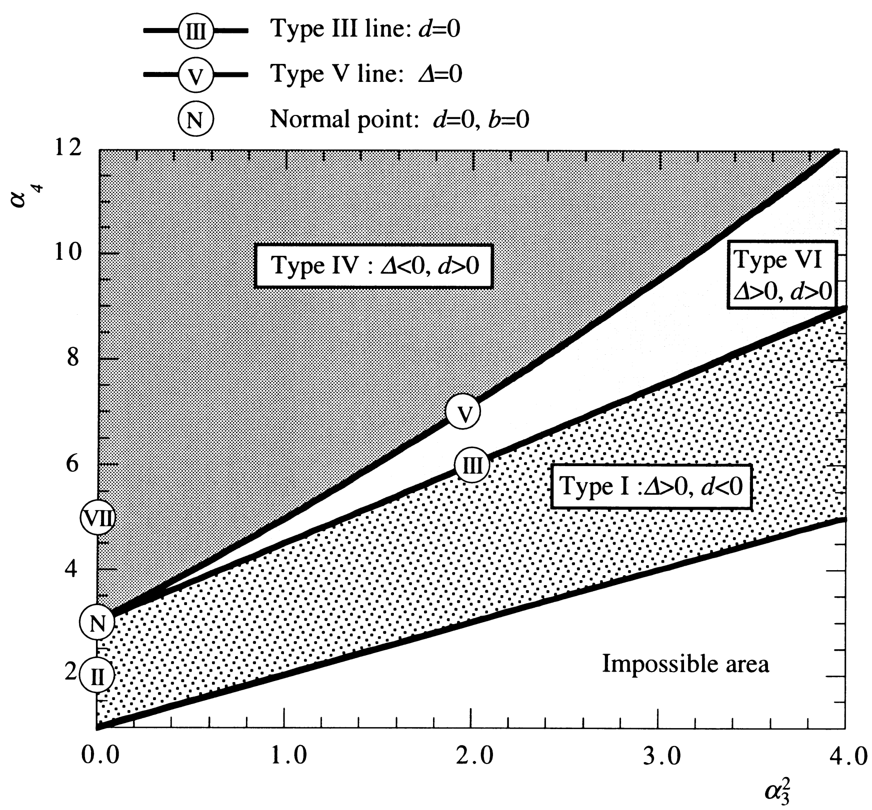

Fig. 10. $\alpha_3^2-\alpha_4$ plane for Pearson system.

References

[1] Madsen HO, Krenk S, Lind NC. Methods of structural safety. Englewood Cliffs (N.J.): Prentice-Hall, 1986.

[2] Bjerager P. Methods for structural reliability computation. In: Casciati F, editor. Reliability problems: general principles and applications in mechanics of solid and structures. New York: Springer Verlag Wien, 1991. p. 89–136.

[3] Hasofer AM, Lind NC. Exact and invariant second-moment code format. J Engrg Mech Division, ASCE 1974;100(1):111–21.

[4] Shinozuka M. Basic analysis of structural safety. J Struct Engrg, ASCE 1983;109(3):721–40.

[5] Rackwitz R. First-order reliability theories and stochastic models. In: Proc. of the 2nd Int. Conf on Struc. Safety and Reliability. Munchen: Werner Verlag, 1977.

[6] Fiessler B, Neumann H-J, Rackwitz R. Quadratic limit states in structural reliability. J Engrg Mech ASCE 1979;105(4):661–76.

[7] Breitung K. Asymptotic approximation for multi-normal integrals. J Engrg Mech ASCE 1984;10(3):357–66.

[8] Koyluoglu HU, Nielsen SRK. New approximations for SORM integrals. Structural Safety 1994;13:235–46.

[9] Der Kiureghian A, Lin HZ, Hwang SJ. Second-order reliability approximations. J Engrg Mech ASCE 1987;113(8):1208–25.

[10] Bucher CG. Adaptive sampling: an iterative fast Monte-Carlo procedure. Structural Safety 1988;5(2):119–26.

[11] Mori Y, Ellingwood B. Time-dependent system reliability analysis by adaptive importance sampling. Structural Safety 1993;12(1):59–73.

[12] Tichy M. First-order third-moment reliability method. Structural Safety 1994;16:189–200.

[13] Bucher CG, Bourgund U. A fast and efficient response surface approach for structural reliability problems. Structural Safety 1990;7:57–66.

[14] Faravelli L. Response surface approach for reliability analysis. J Engrg Mech, ASCE 1989;115(12):2763–81.

[15] Rajashekhar MR, Ellingwood BR. A new look at the response surface approach for reliability analysis. Structural Safety 1993;12:205–20.

[16] Zhao YG, Jiang JR. A structural reliability method based on genetic algorithm. Earthquake Engineering and Engineering Vibration 1995;15(3):44–58 (in Chinese).

[17] Kuschel N, Pieracci A, Rackwitz R. Multiple β-points in structural reliability. In: 3rd Int. Conf. on Computational Stochastic Mechanics, Santorini, 14–17 June 1998.

[18] Der Kiureghian A, Dakessian T. Multiple design points in first- and second-order reliability. Structural Safety 1998;20:37–50.

[19] Stuart A, Ord JK. Kendall's advanced theory of statistics, vol. 1. London: Charles Griffin & Company Ltd, 1987. p. 210–275.

[20] Winterstein SR. Nonlinear vibration models for extremes and fatigue. J Engrg Mech, ASCE 114;1772–1790

[21] Winterstein S, Bjerager P. The use of higher moments in reliability estimation. Proc Int Conf Application of Statistics and Probability 1987;2:1027–36.

[22] Kohno M, Sakamoto J. Multivariate reliability analysis by Hermite polynomial approximation of non-normal distributions. Proc Int Conf Application of Statistics and Probability 1987;2:1123–9.

[23] Parkinson DB. First-order reliability analysis employing translation systems. Engineering Structures 1978;1:31–40.

[24] Grigoriu M, Lind NC. Optimal estimation of convolution integrals. J Engrg Mech, ASCE 1980;106(6):1349–64.

[25] Ono T, Idota H. Development of high-order moment standardization method into structural design and its efficiency. J Structural and Construction Engineering, AIJ 1986;365:40–7 (in Japanese).

[26] Grigoriu M. Approximate analysis of complex reliability problems. Structural Safety 1983;1:277–88.

[27] Hong HP. Point-estimate moment-based reliability analysis. Civil Engineering System 1996;13:281–94.

[28] Pearson ES, Johnson NL, Burr IW. Comparison of the percentage points of distributions with the same first four moments, chosen from eight different systems of frequency curves. Commun Statist-Simula Computa, B 1979;8:191–229.

[29] Slifker JF, Shapiro SS. The Johnson system: selection and parameter estimation. Technometrics 1980;22(2):239–46.

[30] Ramberg JS, Dudewitz EJ, Tadikamalla PR, Mykytka EF. A probability distribution and its uses in fitting data. Technometrics 1979;21:201–14.

[31] Zhao YG, Ono T. New point-estimates for probability moments. J Engrg Mech, ASCE 2000;126(4):433–6.
[32] Abramowitz M, Stegum IE. Handbook of mathematical functions. 10th ed. Dover (New York). 1972. p. 924.
[33] Zhao YG, Ono T. System reliability evaluation of ductile frame structures. J Struct Engrg, ASCE 1998;124(6):678–85.
[34] Ditlevsen O. Narrow reliability bounds for structural systems. J Struct Mech 1979;7:453–72.
[35] Zhao YG, Ono T. New approximations for SORM: part 2. J Engrg Mech, ASCE 1999;125(1):86–93.