System Reliability Assessment by Method of Moments
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Abstract: The computational assessment of system reliability of structures has remained a challenge in the field of reliability engineering. Calculation of the failure probability for a system is generally difficult even if the potential failure modes are known or can be identified, because available analytical methods require determination of the sensitivity of performance functions, information on mutual correlations among potential failure modes, and determination of design points. In the present paper, a method based on moment approximations is proposed for structural system reliability assessment that is applicable to both series and nonseries systems. The point estimate method is applied to evaluate the first few moments of the system performance function of a structure from which the moment-based reliability index and failure probability can be evaluated without Monte Carlo simulations. The procedure does not require the computation of derivatives, nor determination of the design point and computation of mutual correlations among failure modes; thus, it should be computationally effective for structural assessment of system reliability.

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Introduction

The evaluation of system reliability for structures has been an active area of research for over three decades. The calculation of the failure probability for a system is generally difficult even if the potential failure modes are known or can be identified, because of the large number of potential failure modes for most practical structures, the difficulty in obtaining the sensitivity of the performance function, and the mutual correlations among failure modes. The search for efficient computational procedures for estimating system reliability has resulted in several approaches such as bounding techniques, the probabilistic network evaluation technique (PNET), and direct or smart Monte Carlo simulations. In the present paper, a computationally more effective method using moment approximations for system reliability is proposed and examined for both series and nonseries systems.

Assessment of System Reliability

A structural system will invariably have multiple modes of potential failure, e.g., \( E_1, E_2, \ldots, E_m \). The occurrence of one or more of these failure modes will constitute failure of the system, i.e., system failure is the union of all the modes or \( E_1 \cup E_2 \cup \ldots \cup E_m \).

For a structural system, each of the failure modes, \( E_i \), can be defined by a performance function \( g_i = g_i(X) \) such that \( E_i = (g_i < 0) \) and the failure probability of the system is then

\[
P_F = \text{Prob}[g_1 \leq 0 \cup g_2 \leq 0 \cup \ldots \cup g_m \leq 0]
\]

Conversely, the safety of a system is the event in which none of the \( m \) potential failure modes occurs; again in the case of a series system, this means

\[
P_S = \text{Prob}[g_1 > 0 \cap g_2 > 0 \cap \ldots \cap g_m > 0]
\]

Thus the performance function of a series system, \( G \), can be expressed as the minimum of the performance functions that corresponds to all the potential failure modes, that is,

\[
G(X) = \min\{g_1, g_2, \ldots, g_m\}
\]

where \( g_i = g_i(X) \) is the performance function of the \( i \)th failure mode.

In the case of a series system, the performance functions of the individual failure modes will be smooth; for a nonseries system, however, each of the failure modes will generally involve combinations of the maximum and minimum of the component performance functions, as illustrated later in example 4. Consequently, the resulting system performance function, \( G(X) \), will not be smooth and will be more complex than that of a comparable series system.

Since it is difficult to obtain the sensitivity of the performance function even for a series system like Eq. (3), derivative-based FORM would not be applicable. The failure probability of a system can be determined using bounding techniques (see, e.g., Cornell 1966) as a function of the failure probability of individual modes; however, for a complex system the bounds would be wide even though these bounds can be improved by second-order bounds (Ditlevsen 1979). The failure probability of a system may also be estimated approximately with the probabilistic network evaluation technique developed by Ang and Ma (1981), in which mutual correlations among the failure modes have to be computed. Other methods have been reviewed or discussed, e.g., by Moses (1982), Thoft-Christensen and Murotsu (1986), and Bennett and Ang (1986).

In the present paper, moment approximations (Zhao and Ono

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Determining Moments of Performance Function

In the present paper, the point-estimate method (Zhao and Ono 2000a) is used to determine the moments of the system performance function like in Eq. (3) for a series system. For a function of only one random variable \( y = y(x) \), the moments of \( y \) can be point estimated as

\[
\mu_y = \sum_{k=1}^{m} P_k [T^{-1}(u_k)] \quad (4a)
\]

\[
\sigma_y^2 = \sum_{k=1}^{m} P_k [y[T^{-1}(u_k)] - \mu_y]^2 \quad (4b)
\]

\[
\alpha_{y, \sigma_y} \sigma_y^2 = \sum_{k=1}^{m} P_k [y[T^{-1}(u_k)] - \mu_y] \quad (4c)
\]

where \( \mu_y \), \( \sigma_y \), and \( \alpha_{y, \sigma_y} \) are mean value, standard deviation, and \( r \)th dimensionless central moment of \( y(x) \), \( T^{-1} \) is inverse Rosenblatt transformation. \( u_1, u_2, \ldots, u_m \) are \( m \) estimating points and \( P_1, P_2, \ldots, P_m \) are corresponding weights.

The estimating points \( u_i \) and their corresponding weights \( P_i \) can be readily obtained as (Zhao and Ono 2000a)

\[
u_i = \sqrt{2} x_i \quad P_i = \frac{w_i}{\sqrt{\pi}} \quad (5)
\]

where \( x_i \) and \( w_i \) are the abscissas and weights for Hermite integration with weight function \( \exp(-x^2) \) that can be found in work by Abramowitz and Stegun (1972).

For a five point estimate in standard normal space (Zhao and Ono 2000a),

\[
u_0 = 0 \quad P_0 = 8/15 \quad (6a)
\]

\[
u_1 = -u_1 = 1.3556262 \quad P_1 = 0.2220759 \quad (6b)
\]

\[
u_2 = -u_2 = 258.9700 \quad P_2 = 1.12574 \times 10^{-2} \quad (6c)
\]

whereas for a seven point estimate in standard normal space,

\[
u_0 = 0 \quad P_0 = 16/35 \quad (7a)
\]

\[
u_1 = -u_1 = 1.1544054 \quad P_1 = 0.2401233 \quad (7b)
\]

\[
u_2 = -u_2 = 2366.7594 \quad P_2 = 3.07571 \times 10^{-2} \quad (7c)
\]

\[
u_3 = -u_3 = 3.7504397 \quad P_3 = 5.48269 \times 10^{-4} \quad (7d)
\]

For a function of many variables \( Z = G(X) \), where \( X = x_1, x_2, \ldots, x_n \), the joint probability density is assumed to be concentrated at points in the \( m^n \) hyperquadrants of the space defined by the \( n \) random variables, in which \( m \) is the number of estimating points used in the point estimate for functions of respective single random variables. Then the moments of \( Z = G(X) \) can be point estimated as

\[
\mu_G = \sum_{i=1}^{n} P_{ci} G[T^{-1}(u_{c1}, u_{c2}, \ldots, u_{cn})] \quad (8a)
\]

\[
\sigma_G^2 = \sum_{i=1}^{n} P_{ci} [G(T^{-1}(u_{c1}, u_{c2}, \ldots, u_{cn}))-\mu_G]^2 \quad (8b)
\]

\[
\alpha_{y, \sigma_y} \sigma_G^2 = \sum_{i=1}^{n} P_{ci} [G(T^{-1}(u_{c1}, u_{c2}, \ldots, u_{cn}))-\mu_G]^2 \quad (8c)
\]

where \( c \) = distinct combination of \( n \) terms from a group \( \{1, 2, \ldots, m\} \) and \( ci = ith \) term of \( c \). \( u_{ci} = ci \) estimating point and \( P_{ci} = weight \) corresponding to \( u_{ci} \). \( n \) = number of random variables and \( m \) = number of estimating points, where \( \mu_G \), \( \sigma_G \), and \( \alpha_{y, \sigma_y} \sigma_G \) are the mean value, standard deviation, and \( r \)th dimensionless central moment of \( G(X) \), and \( T^{-1} \) is the inverse Rosenblatt transformation.

Since all distinct combinations have to be considered, \( m^n \) times of function calls for computing \( G(X) \) are required. The computations involved in Eqs. (8), therefore, can be massive when \( n \) is large. In order to avoid this problem, the function \( G(X) \) may be approximated by \( G^*(X) \) as follows (Zhao and Ono 2000a):

\[
G^*(X) = \sum_{i=1}^{n} (G_i - G_{\mu}) + G_{\mu} \quad (9a)
\]

where

\[
G_{\mu} = G(\mu) \quad (9b)
\]

\[
G_i = G[T^{-1}(U_i)] \quad (9c)
\]

where \( \mu \) represents the vector in which all the random variables take their mean values, and

\[
U_i = [u_{\mu_1}, u_{\mu_2}, \ldots, u_{\mu_{i-1}}, u_i, u_{\mu_{i+1}}, \ldots, u_{\mu_n}]^T, \quad \text{where } u_{\mu_k}, k = 1, \ldots, n \quad \text{except } i \quad \text{is the } k \text{th value of } u_{\mu_k} \quad \text{which is the vector in } u \text{ space that corresponds to } \mu. \quad G_{\mu} \text{ is a constant and } G_i \text{ is a function of only } u_i.
for specific $G^*(X)$. $T^{-1}$ is the inverse Rosenblatt transformation. For independent random variables $X$, $G$, can simply be expressed as

$$G_i = G(\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_{n+1}, \ldots, \mu_n) \quad (9d)$$

The approximation of Eq. (9a) can be viewed as a generalization of the following observation: If $G(X)$ is of the form

$$G(X) = \sum_{i=1}^{n} a_i x_i \quad \text{or} \quad G(X) = \sum_{i=1}^{n} y_i(x_i) \quad (9e)$$

where $a_i =$ constant; and $y_i = \text{arbitrary function of } x_i$, Eq. (9a) will become exact, i.e., $G^*(X) = G(X)$.

Observe that $u_i$ and $i = 1, \ldots, n$ are independent and $G_i$ is a function only of $u_i$; therefore, $G_i$, $i = 1, \ldots, n$ are also independent. Hence, the first four moments of $G^*(X)$ in Eq. (9a) can be expressed as

$$\mu_G = \sum_{i=1}^{n} (\mu_i - G_i) + G_i \quad (10a)$$

$$\sigma_G^2 = \sum_{i=1}^{n} \sigma_i^2 \quad (10b)$$

$$\alpha_{3G} = \sum_{i=1}^{n} \alpha_{3i} \sigma_i^3 \quad (10c)$$

$$\alpha_{4G} = \sum_{i=1}^{n} \alpha_{4i} \sigma_i^3 + 6 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sigma_i^2 \sigma_j^2 \quad (10d)$$

where $\mu_i$ and $\sigma_i$ = mean value and standard deviation of $G_i$, respectively, $\alpha_{3i}$ and $\alpha_{4i}$ are the third and fourth dimensionless central moments, i.e., the skewness and kurtosis of $G_i$.

Since $G_i$ is a function of only one standard normal random variable $u_i$, the first four moments, $\mu_i$, $\sigma_i$, $\alpha_{3i}$, and $\alpha_{4i}$, can be point estimated from Eqs. (4). For a performance function $G(X)$ with $n$ variables, if the probability moments of $G_i$ are estimated using a $m$-point estimate, only $mn$ function calls of $G(X)$ are required to estimate the first three or four moments of $G(X)$. After the first three or four moments of $G(X)$ are obtained, the reliability analysis becomes a problem of approximating the distribution of a specific random variable with its known first three or four moments.

**Moment-Method Formulas**

**Third-Moment Reliability Index**

The third-moment reliability index can be obtained from some three-parameter distributions, such as the three-parameter lognormal distribution (Tichy 1994) and the third-moment transformation (Zhao and Ono 2000b). These distributions give similar results when the absolute skewness of the performance function is small (e.g., $\alpha_3 < 1$). In the present paper, the three-parameter lognormal distribution is used. For the performance function $z = G(X)$ described in Eq. (3), if the first three moments are obtained, and assuming that the standardized variable $z = \frac{z - \mu_G}{\sigma_G}$ obeys the three-parameter lognormal distribution (Tichy 1994), the standard normal random variable $u$ is expressed as the following function:

$$u = \frac{\text{sign}(\alpha_{3G}) \ln \left( \frac{\sqrt{A} \left( 1 - \frac{z_b}{u_b} \right)}{A} \right)}{\ln(A)} \quad (12)$$

where $\mu_G$ and $\sigma_G = \text{mean and standard deviations of } z$, respectively; and $u_b = \text{standardized bound of the distribution}$. $A = 1 + \frac{1}{\mu_b^2}$ (13)

The relationship between the bound $u_b$ and the skewness $\alpha_{3G}$ is given by

$$\alpha_{3G} = -\left( 3 + \frac{1}{\mu_b^2} \right) \frac{1}{\mu_b} \quad (14)$$

The solution of Eq. (14) yields

$$u_b = (a + b)^{1/3} + (a - b)^{1/3} - \frac{1}{\alpha_{3G}} \quad (15a)$$

in which

$$a = -\frac{1}{\alpha_{3G}} \left( \frac{1}{\alpha_{3G}} + \frac{1}{2} \right) b = -\frac{1}{2\alpha_{3G}} \sqrt{\alpha_{3G}^2 + 4} \quad (15b)$$

Since

$$\text{Prob}[z \leq 0] = \text{Prob}[z_a \leq -\beta_{2M}] \quad (16)$$

the third-moment reliability index (3M reliability index) is (Zhao and Ono 2001)

$$\beta_{3M} = \frac{-\text{sign}(\alpha_{3G}) \left( \exp(\xi^2) \right) + 1}{\sqrt{\ln(1 + \frac{\beta_{2M}}{u_b})}} \quad (17)$$

where $\beta_{2M} = \text{second-moment reliability index (2M reliability index).}$

**Simplification of the 3M Reliability Index**

According to Eq. (12), the absolute value of the random variable $(z_b - u_b)$ obeys the lognormal distribution with parameters $\lambda$ and $\xi$, and the coefficient of skewness $\alpha_{3G}$ is given by

$$\alpha_{3G} = \left( \exp(\xi^2) + 2 \right) \sqrt{\exp(\xi^2) - 1} \quad (18)$$

Comparing Eq. (18) with Eq. (14), one can easily see that

$$u_b = -\text{sign}(\alpha_{3G}) \left( \exp(\xi^2) - 1 \right)^{-1/2} \quad (19)$$

When the absolute value of $\alpha_{3G}$ is less than 1, the absolute value of $\xi$ is less than 0.314, and the following approximation applies within error of less than 2.5%:

$$\sqrt{\exp(\xi^2) - 1} = \xi \quad (20)$$

Substituting Eq. (20) into Eq. (18), one obtains

$$\alpha_{3G} = 3\xi + \xi^3 \quad (21)$$

For small $\xi$ (e.g., $\xi < 0.314$), Eq. (21) yields the following approximations for $\xi$ and $u_b$:

$$u_b = \frac{3}{\alpha_{3G}} \xi = 1/3 \alpha_{3G} \quad (22)$$

Then, the standard normal random variable $u$ can be expressed as follows:

$$u = \frac{\alpha_{3G}}{6} + \frac{3}{\alpha_{3G}} \ln \left( 1 + \frac{1}{3 \alpha_{3G} z_b} \right) \quad (23)$$

and the 3M reliability index becomes.
also agree well with those obtained from Eq. (11),

\[ \beta_{3M} = \frac{-\alpha_{3G}}{6} - \frac{3}{\alpha_{3G}} \ln \left[1 - \frac{1}{3} \alpha_{3G} \beta_{2M} \right] \]  

(24a)

Observe that as \( x \) approaches 0, \( \ln(1+x) = x \), and Eq. (24a) becomes

\[ \beta_{3M} = \beta_{2M} - 1/6 \alpha_{3G} \]  

(24b)

This implies that \( \beta_{3M} \) approaches \( \beta_{2M} \) for extremely small \( \alpha_{3G} \).

For negative \( \alpha_{3G} \), Eq. (24a) is valid for any values of \( \beta_{2M} \). However, for positive \( \alpha_{3G} \), Eq. (24a) is valid only if \( \beta_{2M} < 3/\alpha_{3G} \).

To examine the accuracy of the approximation expressed in Eq. (22), the values of the standardized bound \( u_b \) are depicted in Fig. 1, where the thin solid lines indicate the exact values of \( u_b \) obtained from Eq. (15) and the thick dash lines indicate those obtained with Eq. (22). From Fig. 1, one can see that although Eq. (22) is much simpler than Eq. (15), the \( u_b \) results obtained with Eq. (22) agree well with those obtained from Eq. (15).

The accuracy of the approximate 3M reliability index in Eqs. (24) is also demonstrated in Fig. 2, where the thin solid lines indicate the exact reliability indices obtained with Eq. (17) and the dashed lines indicate those obtained with Eqs. (24). From Fig. 2, one can see that the reliability indices obtained with Eqs. (24) also agree well with those obtained from Eq. (17). One can also see that the 2M reliability index (shown as dash-dotted lines in Fig. 2) contains significant error.

### Fourth-Moment Reliability Index

The fourth-moment reliability index can be obtained utilizing existing systems of frequency curves, such as the Pearson, Johnson, and Burr systems (Stuart and Ord 1987; Hong 1996), and Ramberg’s lambda distribution (Grigoriu 1983). Since the quality of approximating the tail area of a distribution is relatively insensitive to the distribution family selected (Pearson et al. 1979) and the solution of nonlinear equations is necessary to determine the parameters of the Johnson and Burr systems or the lambda distribution, the Pearson system is selected for use in the present study. For the standardized variable \( z_u \) of Eq. (11), the probability density function (PDF) of \( z_u \) and \( f \) satisfies the following differential equation in the Pearson system (Stuart and Ord 1987):

\[
\frac{1}{f} \frac{df}{dz_u} = \frac{az_u + b}{c + bz_u + dz_u^2} \]

(25)

where

\[
a = 10\alpha_{4G} - 12\alpha^2_{3G} - 18
\]

\[
b = \alpha_{3G}(\alpha_{4G} + 3)
\]

\[
c = 4\alpha_{4G} - 3\alpha^2_{3G}
\]

\[
d = 2\alpha_{4G} - 3\alpha^2_{3G} - 6
\]

Using the relationship described in Eq. (16), the fourth-moment reliability index \( 4M \) reliability index) is given by

\[
\beta_{4M} = -\Phi^{-1} \left[ \int_{-\infty}^{-\beta_{2M}} f(z_u) dz_u \right]
\]

(26)

The PDF of \( z_u \), depending on the values of parameters \( a, b, c, \) and \( d \) (Zhao and Ono 2001) is as follows:

\[
f(z_u) = K(z_u - r_2)^{-1/2} (z_u - r_1)^{1/2} (z_u - r_3)^{-1/2} (z_u - r_4)^{1/2}
\]

(27a)

for \( \Delta > 0, \ d < 0 \)

\[
f(z_u) = K(z_u - r_1)^{ac-b^2} \exp \left[ -\frac{az_u}{b} \right]
\]

(27b)

for \( \Delta > 0, \ d = 0 \)

\[
f(z_u) = K(z_u - r_1)^{-1/2} (z_u - r_2)^{-1/2} (z_u - r_3)^{-1/2} (z_u - r_4)^{-1/2}
\]

(27c)

for \( \Delta > 0, \ d > 0 \)

\[
f(z_u) = K(z_u - r_0)^{-a} d^{rb_0} \exp \left[ -\frac{ar_0 + b}{d(z_u - r_0)} \right]
\]

(27d)

for \( \Delta = 0 \)

\[
f(z_u) = K(z_u - r_0)^{-a} d^{rb_0} \exp \left[ -\frac{ar_0 + b}{d(z_u - r_0)} \right]
\]

(27e)

\[
\Delta = b^2 - 4cd
\]

(27f)

One can note that when \( \alpha_{3G} = 0 \) and \( \alpha_{4G} = 3 \), \( z_u \) becomes a standard normal variable; in this case, \( \beta_{4M} = \beta_{2M} \).

## Numerical Examples

A number of examples are illustrated below to demonstrate the accuracy and computational effectiveness of the moment-based method. These include problems that involve normal as well as non-normal probability distributions. If one or more of the random variables are non-normal, appropriate Rosenblatt transforma-
tions are necessary (Ang and Tang 1984) to evaluate the moments as indicated in Eqs. (4) and (8). Most of the examples are series systems with elastoplastic components, however, two nonseries systems are illustrated in example 4 and they involve brittle components. One of the case in example 2 also illustrates the limitation of the first moments for problems that involve extremely small failure probability.

Example 1

In the first example we consider a one-story one-bay elastoplastic frame, shown in Fig. 3(a). This series system is used to illustrate numerical details of the procedure for the proposed moment method. The FORM reliability indices for the respective failure modes of the system can be readily identified and defined by four linear performance functions. The FORM reliability indices for the respective failure modes are given in parentheses below to indicate the relative dominance of the four different modes.

\[
\begin{align*}
G_1(X) &= 2M_1 + M_2 + M_3 - 4.5S \quad (\beta_G = 3.334) \quad (28a) \\
G_2(X) &= 2M_1 + M_2 + 3M_3 - 4.5S \quad (\beta_G = 3.364) \quad (28b) \\
G_3(X) &= M_1 + 2M_2 + 2M_3 - 4.5S \quad (\beta_G = 3.364) \quad (28c) \\
G_4(X) &= M_1 + M_2 + 2M_3 - 4.5S \quad (\beta_G = 3.364) \quad (28d)
\end{align*}
\]

Since this is a series system, the performance function of the system can be defined as the minimum of the above, i.e.,

\[
G(X) = \min\{G_1(X), G_2(X), G_3(X), G_4(X)\} \quad (28e)
\]

where \(M_i\) and \(S\) = independent log--normal random variables with mean and standard deviations of \(\mu_M = 200m\), \(\sigma_M = 50m\), \(\sigma_S = 30m\), and \(\sigma_S = 20r\). Using the mean values of all the variables in Eq. (28e), \(G_\mu\) as defined in Eqs. (9) is

\[
G_\mu = \min\{2 \times 200 + 2 \times 200 - 4.5 \times 50, 2 \times 200 + 200 + 200 \times 200 - 4.5 \times 50, 2 \times 200 + 200 + 200 - 4.5 \times 40\} = \min\{575, 575, 575, 575\} = 575
\]

Substituting the mean values of all the variables except \(M_1\) into Eq. (28e), \(G_1\) as defined in Eqs. (9) becomes

\[
G_1 = \min\{2M_1 + 2 \times 200 - 4.5 \times 50, 2M_1 + 200 + 200 - 4.5 \times 50, M_1 + 200 + 2 \times 200 - 4.5 \times 40, M_1 + 2 \times 200 + 200 - 4.5 \times 40\} = \min\{2M_1 + 175, 2M_1 + 175, M_1 + 375, M_1 + 375\} = \min\{2M_1 + 175, M_1 + 375\}
\]

Similarly, \(G_i\), \(i = 2, 3, 4\), are, respectively,

\[
G_2 = \min\{575, 2M_1 + 175, M_1 + 375\}
\]

\[
G_3 = \min\{2M_1 + 175, M_1 + 375\}
\]

\[
G_4 = 800 - 4.5S
\]

Since \(G_1\) is a function of single random variable \(M_1\), its moments can be point estimated using Eqs. (4). For a five-point estimate, using the inverse Rosenblatt transformation expressed as

\[
T^{-1}(u_i) = F^{-1}[\Phi(u_i)] \quad (29)
\]

where \(F=\) cumulative distribution function of \(M_1\), and \(\Phi=\) standard normal probability, the five estimating points of Eqs. (6) can be easily transformed into original space as \(T^{-1}(u_{i_{\min}}) = 129.157, T^{-1}(u_{i_{\min}}) = 161.576, T^{-1}(u_{i_{\min}}) = 197.787, T^{-1}(u_{i_{\min}}) = 242.113,\) and \(T^{-1}(u_{i_{\min}}) = 302.886\). Substituting these into Eqs. (4) using the corresponding weights listed in Eqs. (6), the first four moments of \(G_1\) are approximately

\[
\mu_1 = 564.489t m, \quad \sigma_1 = 44.164t m, \quad \alpha_31 = -0.479, \quad \alpha_41 = 2.943
\]

Similarly the first four moments of \(G_2, G_3, G_4\) are

\[
\mu_2 = 553.979t m, \quad \sigma_2 = 33.259t m, \quad \alpha_32 = -1.463, \quad \alpha_42 = 3.785
\]

\[
\mu_3 = 564.489t m, \quad \sigma_3 = 44.164t m, \quad \alpha_33 = -0.479, \quad \alpha_43 = 2.943
\]

\[
\mu_4 = 575.000t m, \quad \sigma_4 = 90.000t m, \quad \alpha_34 = -1.264, \quad \alpha_44 = 5.968
\]

Then using Eqs. (10), the first four moments of \(G^*\) are approximately \(\mu_1 = 532.958, \quad \sigma_1 = 114.485, \quad \alpha_31 = -0.704,\) and \(\alpha_41 = 4.098\).

Finally, with the first two moments of the performance function \(G^*,\) the 2M reliability index is \(\beta_{2M} = 4.655.\) With the first three moments of the performance function, Eqs. (24) give the 3M reliability index as \(\beta_{3M} = 3.264\) with corresponding failure probability of \(P_F = 5.498 \times 10^{-4}\). With the first four moments of the performance function, parameters \(a, b, c,\) and \(d\) defined in Eq. (25) are readily obtained to be \(a = 17.030, \quad b = -4.996, \quad c = 14.904,\) and \(d = 0.709.\) Since \(\Delta = b^2 - 4cd = -17.288 < 0,\) the PDF of the standardized performance function \(z_u\) is in the form of Eq. (27e). Using \(F(+\infty) = 1,\) in Eq. (27e) the PDF becomes

\[
f(z_u) = 4263.42(14.904 + 4.996z_u + 0.709z_u^2)^{-12.015}
\]

\[
\times \exp\left[-26.472 \tan^{-1}(0.341z_u - 1.204)\right]
\]

By substituting Eq. (30) and \(\beta_{2M} = 4.655\) into Eq. (26), the 4M reliability index is \(\beta_{4M} = 3.243\) with corresponding failure of \(P_F = 5.91 \times 10^{-4}\).

Using Monte Carlo simulations (MCS) with 1 million samples, the probability of failure for this system is estimated to be \(5.34 \times 10^{-4}\) with a corresponding reliability index of \(\beta = 3.272.\) The coefficient of variation (COV) of this MCS estimate is 4.32%. In this case, only 20 function calls are used (with a total of 20 estimating points for all the variables). For this example, one can see that both the results of the third- and fourth-moment approximations are in close agreement with the MCS results, whereas the second-moment approximation has gross error (42% overestimation of the reliability index).

Example 2

The second example is also a one-story and one-bay elastoplastic frame as shown in Fig. 3(b) (after work by Ono et al. 1990). The statistics of the member strengths and loads are as follows: mean values are \(\mu_{M1} = 500\) ft kip, \(\mu_{M2} = 667\) ft kip, \(\mu_{S1} = \)
For the example, the corresponding $G_i$ of Eqs. (9) is as follows:

\[
G_m = 1250 \text{ ft kip}
\]

\[
G_1 = \min\{917 + M_1, 250 + 2M_1\}
\]

\[
G_2 = \min\{917 + M_2, 250 + 2M_2, 84 + 3M_2\}
\]

\[
G_3 = \min\{1250, 750 + M_3, 250 + 2M_3, -750 + 4M_3\}
\]

\[
G_4 = 2000 - 15S_1
\]

\[
G_5 = \min\{1250, 2584 - 10S_2\}
\]

Using the point estimate method with five estimating points, the first four moments of $G_1$, $G_2$, $G_3$, $G_4$, and $G_5$ are approximately

\[
\mu_1 = 1.248.98 \text{ ft kip}, \quad \sigma_1 = 146.79 \text{ ft kip},
\]

\[
\alpha_{s1} = -0.293, \quad \alpha_{a1} = 2.767,
\]

\[
\mu_2 = 1.248.98 \text{ ft kip}, \quad \sigma_2 = 146.79 \text{ ft kip},
\]

\[
\alpha_{s2} = 0.291, \quad \alpha_{a2} = 2.767,
\]

\[
\mu_3 = 1.246.88 \text{ ft kip}, \quad \sigma_3 = 29.229 \text{ ft kip},
\]

\[
\alpha_{s3} = -9.265, \quad \alpha_{a3} = 88.842,
\]

\[
\mu_4 = 1.250.00 \text{ ft kip}, \quad \sigma_4 = 225.00 \text{ ft kip},
\]

\[
\alpha_{s4} = -0.927, \quad \alpha_{a4} = 4.547.
\]

\[
\mu_5 = 1.250.00 \text{ ft kip}, \quad \sigma_5 = 0.00 \text{ ft kip},
\]

\[
\alpha_{s5} = -0.054, \quad \alpha_{a5} = 3.038.
\]

In this case, $\alpha_5 = 0$, whereas $\alpha_{s5}$ and $\alpha_{a5}$ cannot be obtained according to Eqs. (4). This is because $G_5$ is almost a constant and it has almost no influence on the results of $\sigma_5$, $\alpha_{s5}$, and $\alpha_{a5}$. Any values of $\alpha_{s5}$ and $\alpha_{a5}$ can be used; e.g., $\alpha_{s5} = 0$ and $\alpha_{a5} = 3$, and then use Eqs. (10) as usual, or substitute $G_5$ as a constant in Eqs. (9). The results would remain the same.

Then using Eqs. (10), the first four moments of $G^*$ are approximately $\mu_G = 1.244.85, \sigma_G = 307.523, \alpha_{sG} = -0.307, \alpha_{aG} = 3.426$. With these first four moments of the performance function $G^*$, the $2M$ reliability index is $\beta_{2M} = 4.048$, whereas the $3M$ reliability index of Eqs. (24) is $\beta_{3M} = 3.437$ with corresponding failure probability of $P_F = 2.937 \times 10^{-4}$. The $4M$ reliability index of Eq. (26) gives $\beta_{4M} = 3.276$ with corresponding failure probability of $P_F = 5.268 \times 10^{-4}$. MCS with 1 million samples gives a probability of failure for this system of $6.45 \times 10^{-4}$ with corresponding reliability index of $\beta = 3.218$. The COV of this MCS estimate is 3.94%. For this example, the $3M$ reliability index errors about 6%, whereas the results of the fourth-moment approximation is in close agreement with the MCS results. Again the $2M$ reliability index has a significant error of about 25%.

The reliability analyses for this example (example 2) were extended and different types of distribution of the random variables were assumed. Assuming all the member strengths and loads are Weibull random variables, the results of the moment method and of the MCS with 1 million samples are summarized in column 2 of Table 1. Results for gamma, Gumbel, and normal distributed random variables are also summarized in columns 3–5, respectively, in Table 1. From Table 1, one can observe that irrespective of the types of distribution, both the $3M$ and $4M$ reliability indices are in close agreement with the MCS results. The $2M$ reliability indices, however, consistently contain significant error.

Finally, this example is extended further to examine the applicability (and limitation) to problems with extremely small probability of failure; to do this, the mean loads are assumed to be $\mu_5 = 35 \text{ kip}$ and $\mu_3 = 75 \text{ kip}$. Using the seven-point estimate, the first four moments of $G^*$ are approximately $\mu_G = 1.470.13, \sigma_G = 260.32, \alpha_{sG} = -0.096, \alpha_{aG} = 3.186$. With these first four moments of the system performance function, the $2M$ reliability index is $\beta_{2M} = 5.647$, and the $3M$ reliability index is $\beta_{3M} = 5.207$ with corresponding failure probability of $P_F = 9.58 \times 10^{-8}$. The $4M$ reliability index is found to be $\beta_{4M} = 4.652$ with corresponding failure probability of $P_F = 1.64 \times 10^{-6}$. Using MCS with 30 million samples, the probability of failure for this system is $5.333 \times 10^{-6}$ with corresponding reliability index of $\beta = 4.403$. The COV of this MCS estimate is 7.91%. For this example, the $3M$ and $4M$ reliability indices contain errors of about 18 and 5.6%, respectively, whereas the $2M$ reliability index overestimates the correct value by about 28%. It is interesting to observe that in this case of very small failure probability, the accuracy of the $4M$ reliability index has deteriorated.

### Example 3: Two-Story One-Bay Truss Structure

The third example is an elastoplastic truss structure with two stories and one bay, shown in Fig. 4, which is also a series system. The statistics of the member strengths and loads are as fol-
of each system will fail by tensile fracture or compressive buckling, and thus may be assumed to be brittle, i.e., once failure occurs the strength of a component is reduced to zero. In these two cases, each of the systems is a nonseries system.

For the parallel-chain system shown in Fig. 5(a) there are two failure modes with respect to performance functions of

\[ g_1 = R_1 - S \quad (33a) \]

\[ g_2 = \max \{ \min(R_2, R_3, R_4) - 1/2S, \max(\min(R_2, R_3, R_4) - S) \} \quad (33b) \]

in which fracture strength \( R_i \) and load \( S \) are independent lognormal random variables with means deviations of \( \mu_{R_1} = 2,200 \) kg, \( \mu_{R_2} = 2,100 \) kg, \( \mu_{R_3} = 2,300 \) kg, \( \mu_{R_4} = 2,000 \) kg, and \( \mu_5 = 1,200 \) kg, and standard deviations of \( \sigma_{R_1} = 220 \) kg, \( \sigma_{R_2} = 210 \) kg, \( \sigma_{R_3} = 230 \) kg, \( \sigma_{R_4} = 20 \) kg, and \( \sigma_5 = 240 \) kg.

Similar to, in Fig. 5(a) for the truss in Fig. 5(b), the corresponding performance functions for each of the failure modes can be shown to be those listed below:

\[ g_1 = \max[T_1 - 1.127F_1 + 0.826F_2, \min(T_2 - 4/3F_1, T_3 - F_1 + 3/4F_2, T_4 - 5/4F_1, T_5 - 5/3F_1 + 5/4F_2)] \quad (34a) \]

\[ g_2 = \min(T_1 - 0.206F_1 - 0.826F_2, \min(T_1 - 4/3F_1, T_3 - 3/4F_2, T_4 - 5/3F_1 - 5/4F_2, T_5 - 5/4F_2)] \quad (34b) \]

\[ g_3 = \max[T_3 - 0.155F_1 + 0.13F_2, \min(T_1 - 4/3F_1 + F_1, T_2 - F_2, T_4 - 5/3F_1)] \quad (34c) \]

\[ g_4 = \max[T_4 - 1.409F_1 + 0.217F_2, \min(T_1 - F_2, T_2 - 4/3F_1 - F_2, T_3 - F_1, T_5 - 5/3F_2)] \quad (34d) \]

\[ g_5 = \max[T_5 - 0.258F_1 + 0.217F_2, \min(T_1 - 4/3F_1 + F_2, T_2 - F_2, T_4 - 5/3F_1)] \quad (34e) \]

where fracture strength \( T_i \) and load \( F_i \) are independent lognormal random variables with means and standard deviations of \( \mu_{T_1} = 20 \), \( \mu_{T_2} = 7 \), \( \mu_{T_3} = 20 \), \( \mu_{T_4} = 50 \), \( \mu_{F_1} = 2 \), \( \mu_{F_2} = 21 \), \( \mu_{F_3} = 28 \), \( \mu_{F_4} = 20 \), and \( \sigma_{T_1} = 6 \), \( \sigma_{T_2} = 2 \), \( \sigma_{T_3} = 7 \), \( \sigma_{T_4} = 7 \), \( \sigma_{F_1} = 2.1 \), and \( \sigma_{F_2} = 0.6 \).

Observe that the performance functions of the individual failure modes of both of these nonseries systems involve the maximum and minimum functions of the component’s properties, and therefore are not smooth functions.

**Example 4: Two Brittle Systems**

Illustrated next are two brittle systems that are nonseries systems, a simple parallel-chain system and a truss system, shown in Figs. 5(a and b), respectively. Assume that the individual components of each system will fail by tensile fracture or compressive buckling, and thus may be assumed to be brittle, i.e., once failure occurs the strength of a component is reduced to zero. In these two cases, each of the systems is a nonseries system.

Using the five-point estimate, the first four moments of the system performance function, the moment-based reliability index are \( \beta_{2M} = 4.148 \) and \( \beta_{3M} = 3.356 \) with \( P_F = 3.942 \times 10^{-4} \) and \( \beta_{4M} = 3.229 \) with \( P_F = 6.213 \times 10^{-6} \). Using MCS with 1 million samples, the probability of failure for this system is 7.86 \times 10^{-4} with a corresponding reliability index of \( \beta = 3.161 \). The COV of this MCS estimate is 3.57%. For this example, the 3M MCS reliability index errors about 6%, whereas the 4M reliability index errors about 2%. The 2M reliability index has a significant error of about 30%.

**Fig. 5.** (a) Brittle chain system and (b) brittle truss

**Fig. 4.** Two-story one-bay truss
For Eqs. (33) of the parallel-chain system, the first four moments of $G^*$ are approximately $\mu_G = 993.3599$, $\sigma_G = 316.836$, $\alpha_{3G} = -0.275$, and $\alpha_{4G} = 3.109$. With these the first four moments of the performance function, the moment-based reliability indices are as follows:

\[
\beta_{2M} = 3.135, \\
\beta_{3M} = 2.802 \text{ with } P_F = 2.543 \times 10^{-3}, \\
\beta_{4M} = 2.818 \text{ with } P_F = 2.413 \times 10^{-3}
\]

MCS with 500,000 samples yield the probability of failure for this system as $2.506 \times 10^{-3}$ with a corresponding reliability index of $\beta = 2.806$. The COV of this MCS estimate is 2.82%. One can see that both the results of the third and fourth moment approximations are in close agreement with the MCS results, whereas the second-moment approximation overestimated the reliability index by 12%.

For Eqs. (34) of the truss system, the first four moments of $G^*$ are $\mu_G = 8.972$, $\sigma_G = 3.732$, $\alpha_{3G} = -0.165$, and $\alpha_{4G} = 3.752$. Finally, with these first four moments of the performance function, the moment-based results are as follows:

\[
\beta_{2M} = 2.404 \\
\beta_{3M} = 2.285 \text{ with } P_F = 0.0111 \\
\beta_{4M} = 2.226 \text{ with } P_F = 0.0130
\]

MCS [performed independent of Eq. (34), i.e., by considering all the possible sequences of component failures that can lead to system failure] with 500,000 samples, gives the probability of failure for this system as 0.0139 with a corresponding reliability index of $\beta = 2.199$. The COV of this MCS estimate is 1.19%.

**Example 5: Beam-Cable System**

Consider the simple elastoplastic beam-cable system shown in Fig. 6 (after work by Ang and Tang 1984). The performance functions of the potential failure modes are listed below with the respective FORM reliability indices indicated in parentheses:

\[
g_1 = 6M - L^2/2 \quad (\beta_F = 3.322) \quad (35a) \\
g_2 = F_1L + 2F_2L - 2wL^2 \quad (\beta_F = 3.647) \quad (35b) \\
g_3 = M + F_2L - wL^2/2 \quad (\beta_F = 4.515) \quad (35c) \\
g_4 = 2M + F_1L - wL^2 \quad (\beta_F = 4.515) \quad (35d)
\]

where $M$, $F_1$, $F_2$, and $w$ are normally distributed with mean deviations of $\mu_w = 2$ kip/ft, $\mu_{F_1} = 60$ kip, $\mu_{F_2} = 30$ kip, and $\mu_M = 100$ ft/kip, and COVs of $V_w = 0.2$ and $V_F = V_M = 0.1$.

Using the five-point estimate, the first four moments of $G^*$ are $\mu_G = 293.432$, $\sigma_G = 76.352$, $\alpha_{3G} = -0.574$, and $\alpha_{4G} = 3.265$. Finally, with the first four moments of the performance function, the moment-based results are as follows:

\[
\beta_{2M} = 5.843 \\
\beta_{3M} = 2.976 \text{ with } P_F = 1.460 \times 10^{-3}, \\
\beta_{4M} = 3.170 \text{ with } P_F = 7.634 \times 10^{-4}
\]

MCS with 500,000 samples gives the probability of failure for this system as $5.26 \times 10^{-3}$ with corresponding reliability index of $\beta = 3.276$. The COV of this MCS estimate is 6.16%. One can see that the $2M$ reliability index contains a significant number of errors (17%) and the $3M$ reliability index underestimates the reliability index by 9.2%, whereas the $4M$ reliability index has about 3.2% error which is still large. Correctly, using Eqs. (6) and (8), the five-point estimates for the first four moments are $\mu_G = 297.270$, $\sigma_G = 78.780$, $\alpha_{3G} = -0.2089$, and $\alpha_{4G} = 3.2546$. With these more accurate first four moments of the performance function, the $2M$ reliability index is $\beta_{2M} = 3.733$ and the $3M$ and $4M$ reliability indices are $\beta_{3M} = 3.315$ and $\beta_{4M} = 3.255$. Clearly, with these latter results, the third- and fourth-moment approximations are now in closer agreement with the MCS results. This means that if the first four moments are correctly obtained, the reliability of a system can be computed without only significant error. As illustrated in this example, the approximation of the system performance function using Eqs. (9) and the moments generated with Eqs. (10) may, in rare cases, contain significant error. In this case, Eqs. (7) and (8) may be required in order to obtain more accurate results for the moments.

**Principal Conclusions**

1. A moment-based method for assessing the system reliability of series and nonseries structures was proposed, with emphasis on series systems. The method directly calculates the reliability indices (and associated failure probability) based on the first few moments of the system performance function of a structure. It does not require a reliability analysis of individual failure modes; also, it does not need iterative computation of derivatives, nor computation of mutual correlations among failure modes, and does not require any design points. Thus, the moment method proposed should be more effective for evaluation of the system reliability of complex structures than currently available computational (non-MCS) methods.

2. The method also includes the approximate system performance function, $G^*(X)$ of Eqs. (9) and the first four moments of Eqs. (10), both of which lead to significant simplification of the calculation of system reliability indices.

3. The accuracy of the results obtained with the proposed method was thoroughly examined by comparisons with large sample Monte Carlo simulations. The fourth-moment reliability index for a structural system is invariably close to the corresponding reliability index obtained from large sample Monte Carlo simulations. The error associated with the third-moment reliability index may be acceptable for practical purposes, whereas the second-moment reliability index invariably leads to very significant unacceptable errors.

4. The accuracy of the $4M$ reliability index inferred above is generally limited to problems with not very small failure probabilities. However, for problems that involve extremely small failure probabilities, such as the final case examined in
example 2, the first four moments may not be sufficient; in such cases, higher-order moments would be required for accurate results which would necessarily entail more complicated calculations.

5. There may be occasions, such as that in the case examined in example 5, when the exact system performance function and associated moments of Eqs. (7) and (8) may be required for more accurate results. Nevertheless, for realistic and practical structures, the $3M$ and $4M$ reliability indices obtained by simplifying Eqs. (9) and (10) should be sufficiently accurate for overall assessment of system reliability.

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