A Simple Third-Moment Method for Structural Reliability

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Abstract

The objectives of the present paper are to investigate the applicable range of the third-moment method for structural reliability and to suggest a simple third-moment method for practical application in engineering. The applicable range of the second-moment method is also given. The applicable range of the third-moment method is obtained through investigation of the differences among several third-moment methods. Within the applicable range, it is found that the simple reliability index has a good agreement with the original one, and it is therefore suggested as a simple third-moment reliability index. Since only the first three central moments of the performance functions are used, and since it is unnecessary to know the probability distribution of the basic random variables, the present method should be practical in engineering. In order to investigate the efficiency of the proposed method, several examples are examined under different conditions.

Keywords: second-moment method; third-moment method; applicable range; skewness; FORM

1. Introduction

The fundamental problem in structural reliability theory is the computation of the multi-fold probability integral

$$\begin{aligned} P_f &= P[z = G(\mathbf{X}) \le 0] = \int_{G(\mathbf{X})\le 0} f_{\mathbf{X}}(\mathbf{X})d\mathbf{X} \\ &= \int \cdots \int_{G(\mathbf{X})\le 0} f_{\mathbf{X}}(x_1, \cdots, x_n)dx_1 \cdots dx_n \end{aligned} \tag{1}$$

where $\mathbf{X}=[x_1, ..., x_n]^T$, in which the superscripted T= transpose, is a vector of random variables representing uncertain structural quantities, such as loads, environmental factors, material properties, structural dimensions, and variables introduced to account for modeling and prediction errors. $f_{\mathrm{X}}(\mathbf{X})$ denotes the joint probability density function (PDF) of $\mathbf{X}$. $G(\mathbf{X})$ is the performance function defined such that $G(\mathbf{X})\le 0$, the domain of integration, denotes the failure set, and P_f is the probability of failure.

Difficulty in computing this probability has led to the development of various approximation methods, of which the first-order reliability method (FORM) (Hasofer and Lind, 1974; Rackwitz, 1976; Shinozuka, 1983) is considered to be one of the most acceptable computation methods. Due to the contributions of numerous studies, many reliability methods based on FORM have been developed. These include the second-order reliability method (SORM) (Der Kiureghian *et al*., 1987) and the response surface approach (Faravelli, 1989; Liu and Moses, 1994).

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(Received August 31, 2005 ; accepted December 26, 2005)

It has been reported that several practical problems would be accounted when using FORM and the methods based on FORM (Zhao and Ono, 2000a). First, all the basic random variables are assumed to have a known probability distribution. However, in reality, the probability distributions of random variables are often unknown due to the lack of statistical data. Secondly, the derivative-based iteration has to be used, and the iteration may be endless and the computation process is quite complicated. Thirdly, the problem of multi-design points remains. Therefore, it is important to find a simpler and more effective way to conduct reliability analysis, even when the probability distributions of random variables are unknown.

Recently, a method based on moment approximations was proposed for structural reliability analysis. It is based on another expression of failure probability as follows:

$$P_f = P[z = G(\mathbf{X}) \le 0] = \int_{-\infty}^{0} f_z(z)dz = F_z(0) \tag{2}$$

where $z=G(\mathbf{X})$ is also a random variable with corresponding PDF $f_z(z)$.

According to Eq. 2, the failure probability can be evaluated directly by utilizing the central moments of the performance function. If the central moments of the performance function can be obtained, the failure probability, which is defined as the probability when the performance function is less than or equal to zero, can be expressed as a function of the central moments. By finding the relationship between the failure probability and the central moments, the failure probability can be obtained.

For a performance function $z=G(\mathbf{X})$, without loss of generality, $G(\mathbf{X})$ can be standardized as follows:

$$z_u = (z - \mu_G)/\sigma_G \tag{3}$$

where μ_G and σ_G are the mean value and standard deviation of $G(\mathbf{X})$, respectively. Then Eq. 2 can be expressed as

$$P_f = \int_{-\infty}^{-\beta_{2M}} f_{z_u}(z_u)dz_u \tag{4}$$

where

$$\beta_{2M} = \mu_G/\sigma_G \tag{5a}$$

where β_{2M} is the second-moment (2M) reliability index.

If $z=G(\mathbf{X})$ is a normal random variable, β_{2M} is correct, and the failure probability can be expressed as

$$P_f = \Phi(-\beta_{2M}) \tag{5b}$$

where Φ is the cumulative distribution function (CDF) of a standard normal random variable.

When $z=G(\mathbf{X})$ is a non-normal random variable, the reliability index expressed in Eq. 5 is usually not correct, and the first two moments are inadequate, high-order moments are invariably necessary.

The third-moment (3M) method has been suggested, and several 3M reliability indices have been proposed to approximate the probability of failure (Zhao and Ono, 2001a). However, the applicable range of the 3M method has not been adequately investigated. Further, it is difficult to select a suitable reliability index from the many 3M reliability indices.

The objectives of the present paper are to investigate the applicable range of the 3M method for structural reliability and to suggest a simple 3M method for practical application in engineering. The applicable range of the 2M method is also given. The applicable range of the 3M method is obtained through investigation of the differences among several 3M methods. Within the applicable range, it is found that the simple reliability index has a good agreement with the original one and it is therefore suggested as the simple 3M reliability index. Since only the first three central moments of the performance functions are used, and since it is unnecessary to know the probability distribution of the basic random variables, the method should be practical in engineering. In order to investigate the efficiency of the proposed method, several examples are examined under different conditions.

2. Review of 3M Reliability Indices

For a performance function $z=G(\mathbf{X})$, if the first three moments are obtained, assuming that the standardized variable z_u defined by Eq. 3 obeys three-parameter (3P) distributions (Tichy, 1994; Zhao and Ono, 2000b; Zhao and Ang, 2002), respectively, several 3M reliability indices can be derived.

Assuming that z_u obeys the 3P lognormal distribution (Tichy, 1994), the standard normal random variable u can be expressed as the following function (Zhao and Ono, 2001a)

$$u = \frac{Sign(\alpha_{3G})}{\sqrt{\ln(A)}} \ln\left[\sqrt{A}\left(1 - \frac{z_u}{u_b}\right)\right] \tag{6}$$

where

$$A = 1 + 1/u_b^2 \tag{7a}$$

$$u_b = (a+b)^{1/3} + (a-b)^{1/3} - 1/\alpha_{3G} \tag{7b}$$

$$a = -\left(1/\alpha_{3G}^2 + 1/2\right)/\alpha_{3G}, \quad b = \left(\sqrt{\alpha_{3G}^2 + 4}\right)/2\alpha_{3G}^2 \tag{7c}$$

where α_{3G} is the third dimensionless central moment, i.e., the skewness of $z=G(\mathbf{X})$, $Sign(x)$ gives –1, 0 or 1, depending on whether x is negative, zero or positive.

The 3M reliability index based on Eq. 6 is obtained as

$$\beta_{3M} = \frac{-Sign(\alpha_{3G})}{\sqrt{\ln(A)}} \ln\left[\sqrt{A}\left(1 + \frac{\beta_{2M}}{u_b}\right)\right] \tag{8}$$

where β_{3M} is the 3M reliability index. Here, the reliability index defined by Eq. 8 is referred as $\beta_{3M\text{-}1}$.

Assuming that z_u obeys the 3P square normal distribution (Zhao and Ono, 2000b), u can be expressed as the following function

$$u = \frac{1}{2\lambda}\left(\sqrt{1 + 2\lambda^2 + 4\lambda z_u} - \sqrt{1 - 2\lambda^2}\right) \tag{9}$$

where

$$\lambda = Sign(\alpha_{3G})\sqrt{2}\cos\left[\frac{\pi + |\theta|}{3}\right], \theta = tan^{-1}\left(\frac{\sqrt{8 - \alpha_{3G}^2}}{\alpha_{3G}}\right) \tag{10}$$

The 3M reliability index based on Eq. 9 is obtained as

$$\beta_{3M} = \frac{1}{2\lambda}\left(\sqrt{1 - 2\lambda^2} - \sqrt{1 + 2\lambda^2 - 4\lambda\beta_{2M}}\right) \tag{11}$$

From Eq. 10, α_{3G} should be limited in the range of

$$-2\sqrt{2} \le \alpha_{3G} \le 2\sqrt{2} \tag{12}$$

Hereafter, the reliability index defined by Eq. 11 is referred as $\beta_{3M\text{-}2}$.

According to the definition of the 3P Gamma distribution introduced by Zhao and Ang (2002), the standard form of the CDF of z_u is expressed as

$$F(z_u) = F_{g\lambda^2}[\lambda(\lambda + z_u)] \tag{13}$$

$$\lambda = 2/\alpha_{3G} \tag{14}$$

where F_{g,λ^2} is the CDF of the standard Gamma distribution with parameter λ^2.

The 3M reliability index based on Eq. 13 is obtained as

$$\beta_{3M} = -\Phi^{-1}\left\{F_{g\lambda^2}[\lambda(\lambda - \beta_{2M})]\right\} \tag{15}$$

Here, the reliability index defined by Eq. 15 is referred as $\beta_{3M\text{-}3}$.

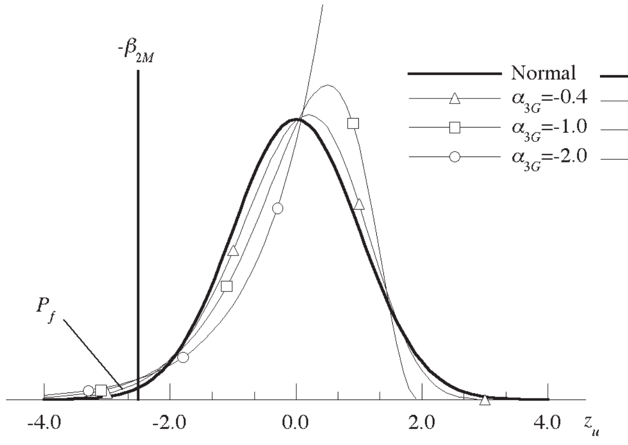


Fig.1. PDF with Negative Skewness

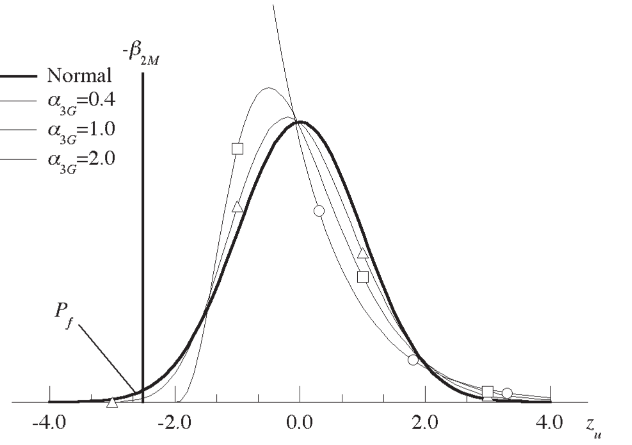
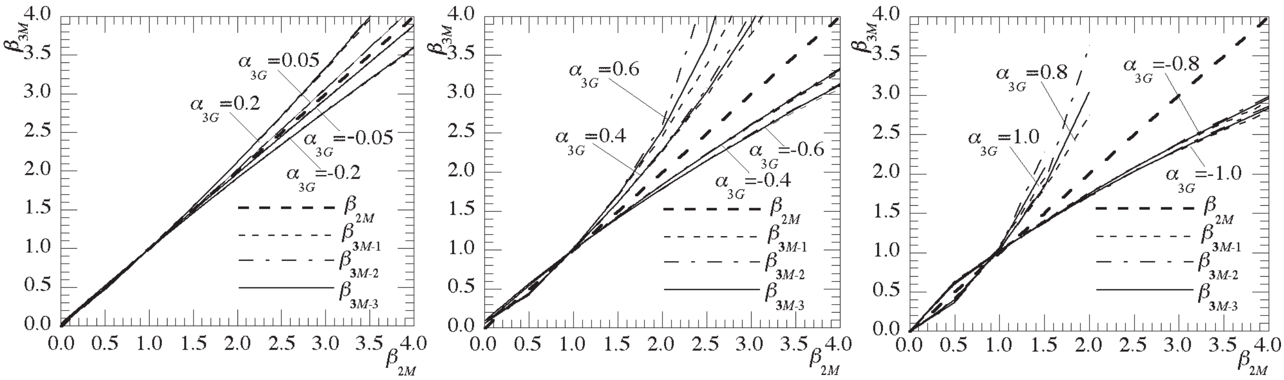


Fig.2. PDF with Positive Skewness



(a) α_{3G} =-0.2, -0.05, 0, 0.05, 0.2 (b) α_{3G} =-0.6, -0.4, 0, 0.4, 0.6 (c) α_{3G} =-1.0, -0.8, 0, 0.8, 1.0

Fig.3. 3M Reliability Indices with Respect to β_{2M}

3. Applicable Range of 3M Method

Obviously, the 3M method is an approximation method, and thus it is expected to have a range of application. The representative PDFs of the 3P distributions are depicted in Figs.1. and 2.. From Figs.1. and 2., one can see that the left tail of the PDF is long for negative α_{3G}, and the right tail is long for positive α_{3G}. Because the failure probability is integrated in the left tail according to Eq. 4, it is easy to understand that the 3M method is more suitable for negative α_{3G} than positive α_{3G}.

The moment reliability index can be obtained using the following equation (Zhao and Ono, 2000b).

$$z_u = S_u(u) = \sum_{j=1}^{k} a_j u^{j-1} \tag{16}$$

where z_u is the standardized performance function defined by Eq. 3, a_j, j=1, ..., k, are deterministic coefficients that are obtained by making the first k central moments of $S_u(u)$ equal to those of z_u, and u is a standard normal variable.

Using Eq. 16, in order to obtain the rth order polynomials of u, the first r+1 moments must be known. That is to say, the first three moments only determine a square polynomial of u. In fact, the above $\beta_{3M\text{-}2}$ is derived from Eq. 16 when r=2. Since it is difficult to approximate a performance function with third power of u using square polynomials of u, the 3M method may not be applicable to a performance function with more than second power of u.

Because a practical reliability problem should have only one solution, all of the 3M reliability indices are expected to give similar results of failure probability for a specific reliability problem. If the relative differences among $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ are beyond the allowable value, it is thought that the 3M method is out of its applicable range. Conversely, the applicable range of the 3M method can be determined by the rule that the relative differences among $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ are below the allowable value.

From Eqs. 8, 11 & 15, one can see that although $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ are based on different probability distributions and described by different forms, they are all functions of β_{2M} and α_{3G}. Thus, the applicable range of the 3M reliability index will be determined using β_{2M} and α_{3G} as parameters.

$\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ changes with respect to β_{2M} are depicted in Fig.3. for α_{3G}=-1.0, -0.8, -0.6, -0.4, -0.2, -0.05, 0, 0.05, 0.2, 0.4, 0.6, 0.8, and 1.0. From Fig.3., one can see that the smaller the β_{2M}, the smaller the differences among the three 3M reliability indices, and all of them become closer to β_{2M} with the decrease of β_{2M}. The smaller the α_{3G}, the smaller the differences among the three 3M reliability indices, and all of them become closer to β_{2M} with the decrease of α_{3G}. One can also see that the differences among the three indices for positive α_{3G} is much larger than those for negative α_{3G}. This is because the 3M method is more suitable for negative α_{3G} than positive α_{3G}, as described earlier. $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ changes with respect to α_{3G} are depicted in Fig.4. for β_{2M} =2.0, 3.0, and 4.0. From Fig.4., again, one can see that for negative α_{3G}, $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ are almost the same, but for positive α_{3G}, $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ are almost the same when α_{3G} is small; however, as α_{3G} becomes larger the differences among $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ become remarkable.

The relative differences among $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ with respect to α_{3G} are depicted in Fig.5. for β_{2M} =2.0, 3.0, and 4.0. The relative difference is given as $r=2(\beta_{Max}-\beta_{Min})/(\beta_{Max}+\beta_{Min})$, where β_{Max} and β_{Min} are the maximum and minimum of $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$ and $\beta_{3M\text{-}3}$, respectively. From Fig.5., one can see that for positive α_{3G}, the larger the α_{3G}, the larger the relative differences among the three 3M reliability indices, but for negative α_{3G}, the variation is irregular.

For practical cases, β_{2M} is generally considered to be not very small, and the discussion in this paper is concentrated on cases of $\beta_{2M}>1$.

For the case of $\alpha_{3G}>0$, the relative differences among $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$ and $\beta_{3M\text{-}3}$ are listed in Table 1. for β_{2M} =2.0, 2.5, 3.0, and 4.0, respectively, corresponding to a certain value of α_{3G}. Using the means of non-linear fit with a large amount of data like Table 1, α_{3G} satisfying the allowable relative difference of r is approximately obtained as

$$\alpha_{3G} \le 40r/\beta_{2M} \tag{17a}$$

For the case of $\alpha_{3G}<0$, the relative differences among $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ are listed in Table 2. for β_{2M}=2.0, 2.5, 3.0, and 4.0, respectively, corresponding to a certain value of α_{3G}. Similarly, using the means of non-linear fit with a large amount of data like Table 2., α_{3G} satisfying the allowable relative difference of r is approximately obtained as

$$\alpha_{3G} \ge -120r/\beta_{2M} \tag{17b}$$

Thus through the above investigation, the applicable range of the 3M method for $\beta_{2M}>1$ is:

$$-120r/\beta_{2M} \le \alpha_{3G} \le 40r/\beta_{2M} \tag{18}$$

Particularly, for r=2%, then

$$-2.4/\beta_{2M} \le \alpha_{3G} \le 0.8/\beta_{2M} \tag{19}$$

For example, if β_{2M}=2.0, the applicable range of the 3M method is $-1.2 \le \alpha_{3G} \le 0.4$, and if β_{2M}=4.0, the applicable range of the 3M method is $-0.6 \le \alpha_{3G} \le 0.2$.

4. Simplification of 3M Reliability Index

The expressions of $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ are all very complex. For obvious reasons, the 3M reliability index for users or designers should be as simple and accurate as possible.

For $-1<\alpha_{3G}<1$, Eq. 10 can be simplified as the following equation with an error of less than 2% (Zhao *et al.*, 2001b).

$$\lambda = \alpha_{3G}/6 \tag{20}$$

Substituting Eq. 20 into Eq. 11

$$\beta_{3M} = \frac{1}{\alpha_{3G}}\left(\sqrt{9-\frac{1}{2}\alpha_{3G}^2} - \sqrt{9+\frac{1}{2}\alpha_{3G}^2 - 6\alpha_{3G}\beta_{2M}}\right) \tag{21}$$

For small $|\alpha_{3G}|$, through the Taylor expansion of the root term, Eq. 21 can be simplified as

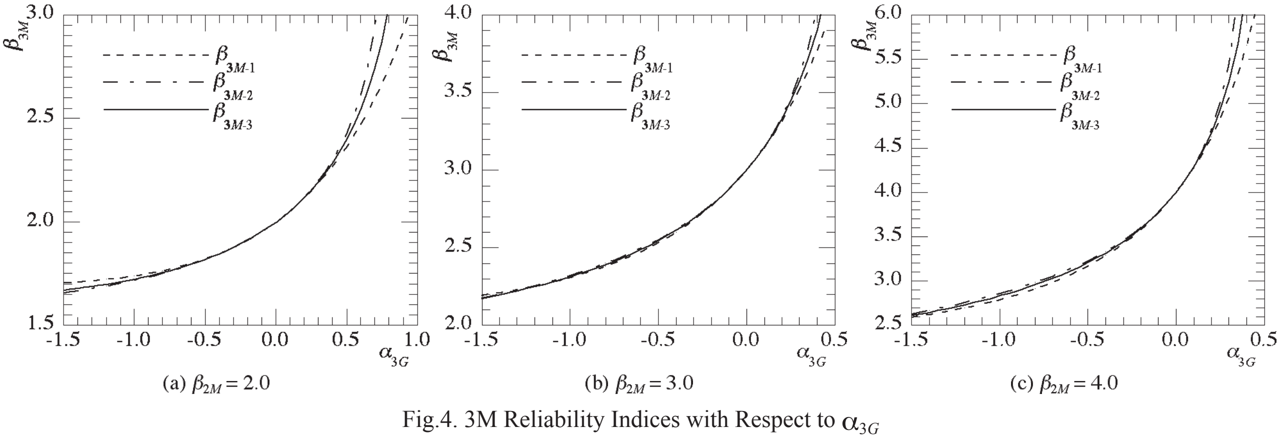
$$\beta_{3M} = \frac{1}{\alpha_{3G}}\left(3 - \sqrt{9+\alpha_{3G}^2 - 6\alpha_{3G}\beta_{2M}}\right) \tag{22}$$

Hereafter, the third moment reliability index defined by Eq. 22 is referred as $\beta_{3M\text{-}4}$.

The comparisons between $\beta_{3M\text{-}2}$ and $\beta_{3M\text{-}4}$ with respect to α_{3G} are shown in Fig.6. for β_{2M} =2.0, 3.0, and 4.0. The applicable range of $\beta_{3M\text{-}4}$ is shown in Fig.7. for an allowable value r=2%. From Figs.6. and 7., one can see that $\beta_{3M\text{-}4}$ approximates $\beta_{3M\text{-}2}$ very well in the applicable range. Thus, $\beta_{3M\text{-}4}$ is the simple 3M reliability index suggested for practical application in engineering.

5. Applicable Range of the 2M Method

It is well known that the 2M method is only suitable for cases in which the performance function $G(\mathbf{X})$ can be approximately expressed by a normal random variable, that is, when the skewness α_{3G} is quite small. However, the applicable range of the 2M method has not been reported according to our knowledge. The problem will be investigated in this section. When $|\alpha_{3G}|$



(a) $\beta_{2M}=2.0$ (b) $\beta_{2M}=3.0$ (c) $\beta_{2M}=4.0$

Fig.4. 3M Reliability Indices with Respect to α_{3G}

Table 1. The Relative Difference among $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ (α_{3G}>0)

β_{2M}	2.0			2.5			3.0			4.0		
α_{3G}	0.30	0.38	0.46	0.21	0.28	0.35	0.17	0.22	0.28	0.12	0.16	0.20
Relative difference	0.5%	1.0%	2.0%	0.5%	1.0%	2.0%	0.5%	1.0%	2.0%	0.5%	1.0%	2.0%

Table 2. The Relative Difference among $\beta_{3M\text{-}1}$, $\beta_{3M\text{-}2}$, and $\beta_{3M\text{-}3}$ (α_{3G}<0)

β_{2M}	2.0			2.5			3.0			4.0		
α_{3G}	-0.72	-0.92	-1.25	-1.05	-1.22	-1.52	-0.32	-0.65	-1.84	-0.18	-0.3	-0.55
Relative difference	0.5%	1.0%	2.0%	0.5%	1.0%	2.0%	0.5%	0.9%	2.0%	0.5%	1.0%	2.0%

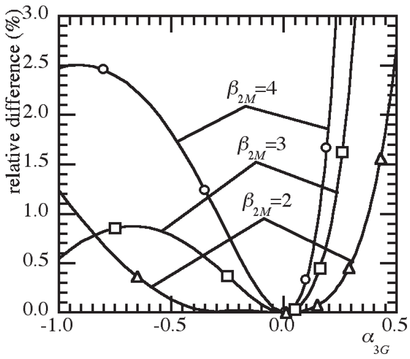


Fig.5. Relative Difference among 3M Indices

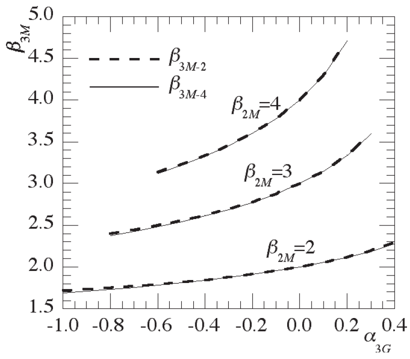


Fig.6. Comparisons between $\beta_{3M\text{-}2}$ and $\beta_{3M\text{-}4}$

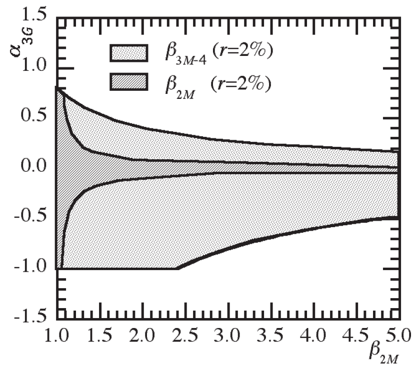


Fig.7. Applicable Range of $\beta_{3M\text{-}4}$ and β_{2M}

is very small, all three β_{3M} can be expressed as (see Appendix A)

$$\beta_{3M} = \beta_{2M} + \frac{1}{6}\alpha_{3G}\left(\beta_{2M}^2 - 1\right) \tag{23}$$

Because β_{2M} is correct only when z=G(**X**) is a nearly normal random variable, i.e., when $|\alpha_{3G}|$ is quite small, Eq. 23 can be used as an accuracy modification of β_{2M}. If the relative errors between β_{3M} and β_{2M} are below the allowable value r, as shown in Eq. 24, it is thought that the 2M method will give good results.

$$\left|\frac{\beta_{3M} - \beta_{2M}}{\beta_{3M}}\right| \le r \tag{24}$$

Substituting Eq. 23 into Eq. 24, finally, for β_{2M}>1,Eq. 24 can be simplified as

$$|\alpha_{3G}| \le \frac{6 \cdot r}{(\beta_{2M} - 1/\beta_{2M})} \tag{25}$$

and then Eq. 25 defines the applicable range of the 2M method. Here, if r is assumed as 2%, then

$$|\alpha_{3G}| \le \frac{0.12}{(\beta_{2M} - 1/\beta_{2M})} \tag{26}$$

The applicable range of the 2M method is also depicted in Fig.7. together with the applicable range of the simple 3M reliability index. From Fig.7., one can see that the applicable range is very small when β_{2M} >2, and when β_{2M} tends to equal to 1.0, the range of α_{3G} tends to equal to that of the 3M method. One can easily understand this from Eq. 23.

6. Numerical Examples

In order to investigate the efficiency of the suggested method, several examples are examined under different conditions.

Example 1.

Consider the following performance function, a plastic collapse mechanism of a one-bay frame, which has been used by Der Kiureghian *et al*. (1987)

$$G(\mathrm{X}) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6 \tag{27}$$

where the variables x_i are mutually independent and lognormally distributed and have means of μ_1=μ_2=μ_3-=μ_4=120, μ_5=50 and μ_6=40, respectively, and standard deviations of σ_1=σ_2=σ_3=σ_4=12, σ_5=15 and σ_6=12, respectively.

Because all the random variables in the above function have a known PDF, the reliability index can be readily obtained using the method of FORM. The FORM reliability index is β_F =2.348, which corresponds to a failure probability of P_f= 0.00943.

The skewness of the variables x_i can be easily obtained as α_{31}=α_{32}=α_{33}=α_{34}=0.301, α_{35}=α_{36}=0.927. The mean value, standard deviation and skewness of G(X) are readily obtained as μ_G=270, σ_G= 103.27, and α_{3G}=-0.528. Using Eq. 5, the 2M reliability index and the corresponding failure probability are readily obtained as β_{2M}=2.615, and P_f =0.00447. Noting that –0.918< α_{3G}=-0.528<0.306, it is in the applicable range of the 3M method. Using Eq. 22, the 3M reliability index is readily obtained as β_{3M}=2.255. The probability of failure corresponding to the 3M reliability index is equal to 0.01207.

The true value of the failure probability is P_f = 0.0121 (Der Kiureghian *et al*., 1987) and the corresponding reliability index is equal to 2.254. Because $|\alpha_{3G}|$ =0.528>>0.0538, the 2M method is significantly in error, and the probability of failure obtained using the proposed method is closer to the true value than that of

FORM.

Example 2.

Consider the following parabolic performance function that was proposed by Der Kiuregian and Dakessian (1998).

where b=5, k=0.5 and e=0.1.

$$G(\mathrm{X}) = b - x_2 - k(x_1 - e)^2 \tag{28}$$

If FORM is used to solve this problem, there are two design points which are successfully obtained by Der Kiuregian and Dakessian (1998) as: X_1*=[-2.741, 0.965]^T with β_1=2.906, and X_2*=[2.916, 1.036]^T with β_2= 3.094.

If the proposed method is used, using point estimates (Zhao and Ono, 2000c), the first three moments of $G(\mathbf{X})$ can be easily obtained as μ_G=4.495, σ_G=1.229, and α_{3G}=-0.555. Using Eq. 5, the 2M reliability index and the corresponding failure probability are readily obtained as β_{2M}=3.657 and P_f= 0.000127, respectively. Using Eq. 22, the 3M reliability index is readily obtained as β_{3M}=2.947. The probability of failure corresponding to the 3M reliability index is equal to 0.001604.

The reliability index using Monte-Carlo Simulation (MCS) obtained by Der Kiuregian and Dakessian (1998) is β=2.751, and the corresponding failure probability is 0.00297. One can see that although the present method does not require the derivative-based iteration and does not use the multiple design points, it provides comparable result of the two first-order reliability indices. However, the result of the present method still has a relative error of 6.88% with the MCS result. This may be because the first three moments are inadequate for such a problem with strong non-normality (Zhao and Ono, 2001c).

Example 3.

Consider the following performance function of a simple structural column (Zhao *et al*., 2001b).

$$G(\mathrm{X}) = Ax_1x_2 - x_3 \tag{29}$$

where A is the nominal section area, x_1 is a random variable representing the uncertainty of A, x_2 is the yield stress, and x_3 is the compressive stress. Assume the column has an H-shape and is made of structural steel with a H300×200 (JIS 1977) section and having an area A=72.38cm^2, and a material of SS41 (JIS 1976). The CDFs of x_1 and x_2 are unknown, and the only information about them are their first three moments: μ_1=0.990,σ_1=0.051, α_{31}=0.709, and μ_2=3.055t/cm^2, σ_2 =0.364,α_{32}=0.512. x_3 is assumed as a lognormal variable with a mean value of μ_3=150t, standard deviation of α_3=45, and skewness α_{33}=0.927.

Because the first three moments of x_1, x_2, and x_3 are known, the first three moments of $G(\mathbf{X})$ can be easily obtained as μ_G=68.910, σ_G=53.238, and α_{3G}=-0.476. Using Eq. 5, the 2M reliability index and the corresponding failure probability are readily obtained as β_{2M}=1.294, and P_f=0.0978. Noting that -1<α_{3G}=-0.476<0.618, it is in the applicable range of the 3M method. With the aid of Eq. 22, the 3M reliability index is readily obtained as β_{3M}=1.250. The corresponding probability of failure is equal to 0.1057.

Because the first three moments are known, the random sampling of x_1 and x_2 can be easily generated

$$x = \sigma\left(-\frac{1}{6}\alpha_3 + \frac{1}{3}\sqrt{9 - \frac{1}{2}\alpha_3^2 u + \frac{1}{6}\alpha_3 u^2}\right) + \mu \tag{30}$$

using Eq. 30 (Zhao *et al*., 2001b)

Thus, MCS can be easily conducted, and the reliability index is equal to 1.275 when the number of samplings is taken to be 10,000. One can see the 3M method is in close agreement with the MCS result. Although $|\alpha_{3G}|$=0.476>0.23, it is out of the applicable range when the allowable value r is assumed as 2%, the 2M method also gives a good result. The reason, as described previously, is that β_{2M} always tends to equal to β_{3M} when β_{2M} tends to equal to1.0.

Example 4.

The fourth example considers a simple reliability problem, shown in Table 3., in which both R and S are lognormal variables with mean value, standard deviation, and skewness of μ_R=175, σ_R=17.5, α_{3R}= 0.301, μ_S=100, σ_S=20, and α_{3S}=0.608. Because both R and S are positive, the five performance functions listed in Table 3. are equivalent. The first three central moments of the performance functions obtained using the seven-point estimates (Zhao and Ono, 2000c) are listed in Table 3. with the results of the 2M and the 3M reliability indices.

From Table 3., one can see that the 3M method is insensitive to the different formulations in the applicable range. For Case 4 and Case 5, the results of the 3M method are significantly in error. This is because the reformulations of the performance function make the skewness exceed the applicable range of the 3M method. Therefore, the insensitivity of the 3M method to the formulation of the limit-states should be limited in the applicable range.

In contrast, the 2M method is very different for the different formulations, and it gives an exact result only in Case 2 because its skewness is in the applicable range. For the other cases, the 2M reliability indices are significantly in error because they all exceed the applicable range. As for FORM, it has almost the same results for the different formulations with the value of β_F=2.590 and it gives good results for this example.

Example 5.

In order to investigate the effect of the probability distribution of random variables, the fifth example considers the following performance function, which is an elementary reliability model that is used in many situations:

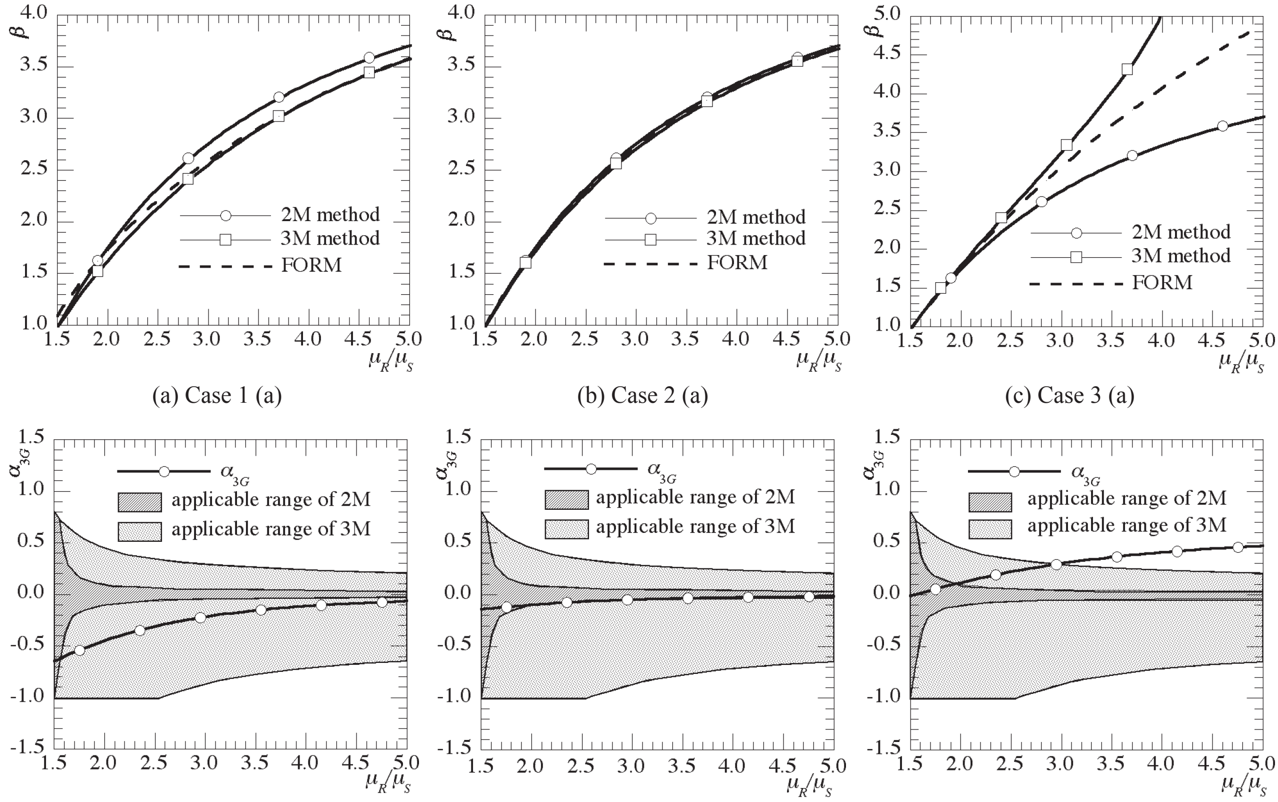
$$G(\mathrm{X}) = R - S \tag{31}$$

where R is resistance and S is load.

Because only two basic random variables are

Table 3. Formula Insensitivity of the 3M Method

G(**X**)		μ_G	σ_G	α_{3G}	β_{2M}	Applicable range of 2M	β_{3M}	Applicable range of 3M
Case 1	R - S	75	26.575	-0.173	2.822	(-0.049, 0.049)	2.649	(-0.850, 0.283)
Case 2	ln$R$ - ln$S$	0.574	0.222	-5.86×10^{-7}	2.590	(-0.054, 0.054)	2.590	(-0.927, 0.309)
Case 3	1 - S/R	0.423	0.130	-0.685	3.264	(-0.041, 0.041)	2.604	(-0.735, 0.245)
Case 4	S/R -1	0.82	0.409	0.685	2.007	(-0.080, 0.080)	2.766	(-1.0, 0.399)
Case 5	$1/S$ - $1/R$	4.63×10^{-3}	2.16×10^{-3}	0.538	2.144	(-0.072, 0.072)	2.717	(-1.0, 0.373)



(a) Case 1 (a) (b) Case 2 (a) (c) Case 3 (a)

(d) Case 1 (b) (e) Case 2 (b) (f) Case 1 (b)

Fig.8. Figures for Ex. 5

involved in Eq. 31 and the expression is a linear function, FORM generally gives good results for this performance function (as shown in Example 4). The first three moments of this performance function can be easily obtained due to the simplicity of this function.

In the following investigations, the coefficient of variation of R is taken to be 0.2 and that of S is taken to be 0.4. The following three cases are investigated under the assumption that R and S obey different probability distributions.

Case 1, R is normal with α_{3R}=0.0 and S is lognormal with α_{3S}=1.264.

Case 2, R is normal with α_{3R}=0.0 and S is Weibull with α_{3S}= 0.2768.

Case 3, R is lognormal with α_{3R}=0.608 and S is Weibull with α_{3S}= 0.2768.

For Cases 1 to 3, the variations of the reliability indices, the skewness, and the applicable range of the 2M and 3M method with respect to μ_R/μ_S (the means of R and S, respectively) are shown in Fig.8.(a)-(f), respectively.

From Fig.8.(a) and (d), one can see that the results of the 3M method for Case 1 are in close agreement with those of FORM in the whole investigation range because α_{3G} is always in the applicable range. The 2M method, meanwhile, gives a good approximation for the results of FORM when μ_R/μ_S is small and has moderately significant errors when μ_R/μ_S is large due to the skewness exceeding the applicable range.

For Case 2, Fig.8.(b) and (e) shows that the results of both the 2M and 3M methods are in close agreement with those of FORM in the whole investigation range since α_{3G} is in the applicable range.

For Case 3, one can see from Fig.8.(c) and (f) that the results of both the 2M and 3M methods agree very well with the FORM results when μ_R/μ_S is small. When μ_R/μ_S is large, the 3M method has significant errors (especially when μ_R/μ_S>3.0) due to α_{3G} being out of the applicable range, and the 2M also produces significant errors when μ_R/μ_S>2.0, and the reason is as described before.

7. Conclusions

The applicable range of the 2M and 3M methods

are determined, and a simple 3M reliability index is suggested to conduct structural reliability analysis in engineering. It is found that:
1. The probability of failure can be computed by using the 2M method or the proposed 3M method, even when the CDFs or PDFs of random variables are unknown.
2. The 3M method is insensitive to the formulations of the limit-states function within its applicable range.
3. The 3M method is more suitable for negative α_{3G} than for positive α_{3G}.
4. There are no significant effects on the accuracy of the proposed 3M method and the 2M method for the different probability distributions of random variables within their range of application.
5. Within the applicable range of the 2M and 3M methods, the two methods usually give good results for reliability evaluations, while for the cases out of their applicable range, the first two or three moments are inadequate, and much higher-order moments are invariably necessary.
6. The 3M method is generally inapplicable to a performance function with more than second power random variables.

Acknowledgements

The study was partially supported by Grant-in-aid from Ministry of ESCS. Japan (No. 17560501). The support is gratefully acknowledged.

APPENDIX A

(1) Simplification of $\beta_{3M\text{-}1}$

For very small $|\alpha_{3G}|$, with aid of a second-order Taylor expansion of log-function $\ln(1+x)$, $\beta_{3M\text{-}1}$ can be rewritten as

$$\beta_{3M-1} = -\frac{\alpha_{3G}}{6} - \frac{3}{\alpha_{3G}}\ln\left(1-\frac{1}{3}\alpha_{3G}\beta_{2M}\right)$$

$$= -\frac{\alpha_{3G}}{6} - \frac{3}{\alpha_{3G}}\left[-\frac{\alpha_{3G}}{3}\beta_{2M} - \frac{1}{2}\left(-\frac{1}{3}\alpha_{3G}\beta_{2M}\right)^2\right]$$

$$= \beta_{2M} + \frac{1}{6}\alpha_{3G}(\beta_{2M}^2 - 1) \qquad \text{(A-1)}$$

(2) Simplification of $\beta_{3M\text{-}2}$

For very small $|\alpha_{3G}|$, with aid of a second-order Taylor expansion of $\sqrt{1+x}$, $\beta_{3M\text{-}2}$ can be expressed as

$$\beta_{3M-2} = \frac{3}{\alpha_{3G}}\left(1-\sqrt{1+\frac{1}{9}\alpha_{3G}^2 - \frac{2}{3}\alpha_{3G}\beta_{2M}}\right)$$

$$= \frac{3}{\alpha_{3G}}\left\{1-\left[1+\frac{1}{2}\times\frac{\left(\alpha_{3G}^2 - 6\alpha_{3G}\beta_{2M}\right)}{9} - \frac{1}{8}\times\frac{\left(\alpha_{3G}^2 - 6\alpha_{3G}\beta_{2M}\right)^2}{9^2}\right]\right\}$$

$$= \beta_{2M} + \frac{1}{6}\alpha_{3G}\left(\beta_{2M}^2 - 1\right) \qquad \text{(A-2)}$$

(3) Simplification of $\beta_{3M\text{-}3}$

Because $\alpha_{3G}\to 0$, $\lambda\to\infty$, and the distribution tends to normality. The standard normal variable u can be expressed as a polynomial function of z_u as the following equation with aid of the Cornish-Fisher expansion (Stuart and Ord, 1987)

$$u = z_u - \frac{1}{6}\alpha_{3G}\left(z_u^2 - 1\right) \qquad \text{(A-3)}$$

thus

$$\beta_{3M-3} = \beta_{2M} + \frac{1}{6}\alpha_{3G}(\beta_{2M}^2 - 1) \qquad \text{(A-4)}$$

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