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On the problems of the Fourth moment method☆

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1. Introduction

This is the response to Xu and Cheng's discussion [1] on the moment methods for structural reliability [2]. In the discussion, the discussers provided some examples that are not applicable by the fourth moment method and also presented some "equivalent" performance functions to demonstrate their criticisms of the method. First of all, the writers appreciate the discussers' interests on the moment reliability method and agree with the discussers that "the moment method has its drawbacks" or limitations. In order to clarify the applicability of the moment method, the problems claimed by the discussers are investigated and it is demonstrated that all the so called equivalent performance functions claimed by the discussers are, in fact, not equivalent, and therefore can not be used to check the validity of the method. It is also demonstrated that all the examples provided by the discussers are out of the application range of the fourth moment method.

2. The applicability of the moment methods

It should be noted that the fourth moment method is an approximate method and it is not strange that it should have an application range. As has been shown in the paper [2], the Pearson system is not the only selection for a moment reliability index. We have some other selections such as the Johnson and Burr systems, and Ramberg's Lamda distribution; many investigations on the applications of these systems can be found in Grigoriu [3], Parkinson [4] and Hong [5]. Since a practical reliability problem should have only one solution, all of these distributions are

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expected to give similar results of failure probability for a specific reliability problem. As indicated in the comparisons of the percentage points of different systems of frequency curves conducted by Pearson et al. [6], the quality of approximating the tail area of a distribution using its first four moments depends on the tail area and the values of the skewness α_3 and kurtosis α_4, and remarkable consistency in percentage points have been observed over considerable regions of the α_3–α_4 plane. This implies that the fourth moment method is applicable over wide ranges of α_3 and α_4. However, if the α_3–α_4 point is far from that of a normal curve, it will be difficult to find a solution in which we can have confidence [6]; therefore, the fourth moment method may not be applicable to a problem with extremely strong non-normality. From Table 1 of the discussion [1], one can see that the skewness of the examples illustrated in the discussion ranged from 4.2 to 6.7 while the kurtosis ranged from 34.8 to 86.6. Comparing these with Pearson's investigation for ($\alpha_3 = 0 \sim 3$ and $\alpha_4 = 3 \sim 14$), one can see that the α_3–α_4 points corresponding to the examples in the discussion are obviously too far from the point of normal curve ($\alpha_3 = 0$, $\alpha_4 = 3$); hence, the problems discussed may not be approached using only the first four moments.

Another selection of the moment reliability index can be obtained by approximating the performance function with its first few moments using the following equation [7]

$$x_s = S_u(u) = \sum_{j=1}^{k} a_j u^{j-1} \tag{1}$$

where x_s is the standardized performance function, a_j, $j = 1, \ldots, k$, are deterministic coefficients that are obtained by making the first k central moment of $S_u(u)$ to be equal to that of x_s.

Using Eq. (1), in order to obtain the rth order polynomials of standard normal variable u, the first $r+1$ moments have to be known. That is to say, the first four moments only determine a cubic polynomial of u, and in order to obtain higher order polynomials of u, more moments have to be used. Since it is difficult to approximate a performance function with fourth power of u using cubic polynomials of u, the fourth moment method may not be applicable to a performance function with fourth power of u. Noting that all the unsuccessful examples claimed by the discussers are performance functions in fourth or fifth powers of u, one can understand why they can not be applied by the fourth moment method. It is obvious that examples claimed by the discussers should be approximated by the first five or six moments of the performance function.

Table 1
Formula insensitivity of the fourth moment reliability index

G(X)	μ_G	σ_G	α_{3G}	α_{4G}	β_{2M}	β_{4M}
R–S	200	36.06	0.070	3.14	5.55	5.17
LnR–ln$S$	1.11	0.22	-1.13×10^{-6}	3.00	5.02	5.02
1–S/R	0.66	0.07	−0.69	3.85	8.77	5.00
R/S–1	2.12	0.70	0.69	3.85	3.02	5.12
1/S–1/R	6.63×10^{-3}	3.34×10^{-4}	−0.301	3.16	3.34	5.19

3. The formula insensitivity of the fourth moment method

In the discussion, the discussers give some performance functions such as the following Eqs. (2a) and (2b) to demonstrate the inapplicability of the fourth moment methods,

$$z = x_1^3 + x_2^3 - 18 \tag{2a}$$

$$z = x_1^2 + x_2^3/x_1 - 18/x_1 \tag{2b}$$

The discussers claimed that Eqs. (2a) and (2b) are equivalent and thus should yield the same reliability analysis results. However, as the matter of fact, the two performance functions are not equivalent. According to the definition of a performance function, the failure region corresponding to both Eqs. (2a) and (2b) should be $z<0$ and needless to say, this definition is independent of the values of the random variables. In Eqs. (2a) and (2b), since x_1 and x_2 are normal random variables as claimed by the discussers, both positive and negative values are possible. For negative values of x_1, the reformulation of performance function from Eqs. (2a) to (2b) will lead to a failure region of $z>0$ and for positive values of x_1, it will lead to a failure region of $z<0$. This is to say, the two performance functions are apparently not equivalent. Using Monte Carlo Simulation with 100,000 samples, the probability of failure corresponding to Eq. (2a) is obtained as 0.00538 (with $\beta=2.5504$) while that corresponding to Eq. (2b) is obtained as 0.02407 (with $\beta=1.9761$). Clearly, the results corresponding to the two performance functions are different. Similarly, all the other so called equivalent performance functions claimed by the discussers, such as Eqs. (8) and (8a), Table 2 in the discussion [1], are not equivalent either and therefore can not be used to check the formula variance of the fourth moment method.

Since the fourth moment method is an approximate method, the application of the method should be limited to its applicable range. A typical example corresponding to this issue is shown in Table 1, in which both R and S are lognormal variables with mean value and standard deviation of $\mu_R=300$, $\sigma_R=30$, $\mu_S=100$, $\sigma_S=20$. Since all the values of both R and S are positive, the five performance functions listed in Table 1 are equivalent. The first four central moments of the performance functions are listed in Table 1 with the results of the second- and fourth-moment reliability indices.

From Table 1, one can see that although the second moment reliability index is much different with the different formulations, the fourth moment reliability index is insensitive to the formulations. However, since the first four moments are quite sensitive to the formulations of the limit state, it is imaginable that the reformulation of the performance function may make the skewness and the kurtosis exceed the applicable range of the fourth moment method. Therefore, the insensitivity of the fourth moment method to the formulation should be limited to the applicable range of the method.

4. The point estimate for probabilistic moment

Although the new point estimate [8] gives much improvement upon the existing point estimates, it should be noted that it is an estimate method, rather than an accurate or a perfect method. For a good behaved function of only one random variable, it has been theoretically varified that the method will approach the accurate results with the increase of the estimating points [9], but for a function of multi-variables, the method sometimes may not approach accurate results [9] since approximation of performance function was used. As has been shown in Ref. [8], the moments of a function of multi-variables can be directly point estimated by concentrating the joint probability density at points in the m^n hyperquadrants of the space defined by the n random variables. Here, m is the number of estimating points. Particully, for a function of only two variables such as that used in the discussion [1], the mean of $Z = G(X)$ can be point-estimated as,

$$\mu_G = \sum_{i=1}^{m}\sum_{j=1}^{m} P_{1i}P_{2j}G\left[T^{-1}\left(u_{1i}, u_{2j}\right)\right] \tag{3}$$

where u_{1i} is the ith estimating point and P_{1i} is the weight corresponding to u_{1i}. Using formulas like Eq. (3) with 7 estimating points for each variable, the first four moments of the discussers' performance function $G = 1 - F/5b^4$ are obtained as $\mu_G = 0.9821$, $\sigma_G = 0.0637$, $\alpha_{3G} = -35.42$, $\alpha_{4G} = 1419.61$, one can see that the results are quit close to the results of MCS provided by the discussers. However, since such computations become massive when n is large, further study is required to develop effective estimates for the first few moments of an arbitrary performance function.

5. Conclusions

1. The fourth moment method should not be applied to performance functions with more than fourth power of normal variable. All the examples demonstrated by the discussers are out of the application range of the fourth moment method.
2. All the so called equivalent performance functions claimed by the discussers are in fact not equivalent and therefore can not be used to invalidate the fourth moment method. In the applicable range of the fourth moment method, the fourth moment reliability index is not sensitive to the formulations of the performance function.
3. Further study is required to develop effective estimate for the first few moments of an arbitrary performance function.

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