New Approximations for SORM: Part 1

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ABSTRACT: In the second-order reliability method the principal curvatures, which are defined as the eigenvalues of rotational transformed Hessian matrix, are used to construct a paraboloid approximation of the limit state surface and compute a second-order estimate of the failure probability. In this paper, the accuracy of the previous formulas of SORM are examined for a large range of not only curvatures but also number of random variables and first-order reliability indices. For easy practical application of SORM in engineering, a simple approximation of SORM is suggested and an empirical second-order reliability index is proposed. By the new approximations, SORM can be easily applied without rotational transformation and eigenvalue analysis of Hessian matrixes. The empirical reliability index proposed in this paper is shown to be simple and accurate among the existing SORM formulas with closed forms. The proposed empirical reliability index gives good approximations of exact results for a large range of curvature radii, the number of random variables, and the first-order reliability indices.

INTRODUCTION

The ultimate goal in structural reliability analysis is to evaluate the probability content of part of an *n*-dimensional probability space. Difficulty in computing this probability has led to the development of various approximation methods, of which the first-order reliability method (FORM) is considered to be one of the most acceptable computational methods (Bjerager 1991). FORM is an analytical approximation in which the reliability index is interpreted as the minimum distance from the origin to the limit state surface in standardized normal space and the most likely failure point (design point) is searched using mathematical programming methods (Shinozuka 1983). Because the performance function is approximated by a linear function at the design point, accuracy problems arise when the performance function is strongly nonlinear.

The second-order reliability method (SORM) has been established as an attempt to improve the accuracy of FORM. SORM is obtained by approximating the limit state surface at the design point by a second-order surface, and the failure probability is given as the probability content outside the second-order surface.

The first thorough study on SORM was performed by Fiessler et al. (1979), in which second-order Taylor expansions as well as curvature-fitted second-order surfaces were considered. An asymptotically exact result for parabolas was derived by Breitung (1984), and a more accurate three-term formula has been proposed by Tvedt (1983). Exact results for a paraboloid were derived by Tvedt (1988) and have been extended to cover all the quadratic forms of Gaussian variables (Tvedt 1990). Furthermore, a point-fitted parabolic algorithm was developed by Der Kiureghian et al. (1987, 1991) and an importance sampling improvement was introduced by Hohenbichler et al. (1988). Recently, new approximations have been obtained using McLaurin series expansion and Taylor series expansion (Koyluoglu and Nielsen 1994; Cai and Elishakoff 1994). However, the applications of these formulas have not been adequately investigated.

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The object of this paper is to investigate the accuracy of previous SORM formulas and suggest a simplified SORM for practical application in engineering. Weaknesses in the current SORM formulas are examined through several examples for a large range of not only curvatures but also number of random variables and first-order reliability indices. A simple approximation of SORM is suggested and an empirical second-order reliability index is proposed.

REVIEW OF SECOND-ORDER RELIABILITY METHOD

The second-order Taylor expansion of a performance function in standardized space $G(\mathbf{U})$ at design point \mathbf{U}^* can be expressed as (Fiessler et al. 1979; Tvedt 1990)

$$G(\mathbf{U}) = \boldsymbol{\beta}_F - \boldsymbol{\alpha}^T \mathbf{U} + \frac{1}{2} \left(\mathbf{U} - \mathbf{U}^* \right)^T \mathbf{B} (\mathbf{U} - \mathbf{U}^*)$$
(1)

where

$$\boldsymbol{lpha} = rac{
abla G(\mathbf{U}^*)}{|
abla G(\mathbf{U}^*)|}; \quad \mathbf{B} = rac{
abla^2 G(\mathbf{U}^*)}{|
abla G(\mathbf{U}^*)|}; \quad eta_F = \boldsymbol{lpha}^T \mathbf{U}^*$$

 β_F = first-order reliability index.

By a rotation of **U** into a new set of mutually independent, standard, normal random variables $\mathbf{X} = \mathbf{HU}$, where the *n*th row of the rotation matrix **H** is $\boldsymbol{\alpha}$, one obtains

$$G'(\mathbf{X}) = -(x_n - \beta_F) + \frac{1}{2} \begin{pmatrix} \mathbf{X}' \\ x_n - \beta_F \end{pmatrix}^T \mathbf{A} \begin{pmatrix} \mathbf{X}' \\ x_n - \beta_F \end{pmatrix}$$
(2)

where $\mathbf{X}' = (x_1, x_2, ..., x_{n-1})$ and $\mathbf{A} = \mathbf{HBH}^T$.

The performance function of (2) can be expressed with no loss of generality by

$$G(\mathbf{Y}) = a + \sum_{j=1}^{n} (\gamma_j y_j + \lambda_j y_j^2)$$
(3)

The exact integral expression for the probability content of the quadratic set is given by Tvedt (1990); simpler expressions were given using the following parabolic surface (Breitung 1984; Tvedt 1983; Der Kiureghian et al. 1987, 1991; Hohenbichler et al. 1988; Koyluoglu and Nielsen 1994; Cai and Elishakoff 1994):

$$G(\mathbf{Y}) = -(y_n - \beta_F) + \frac{1}{2} \sum_{j=1}^{n-1} k_j y_j^2$$
(4)

which implies that (2) is approximated by

$$G'(\mathbf{X}) = -(x_n - \beta_F) + \frac{1}{2} \mathbf{X}'^T \mathbf{A}' \mathbf{X}'$$
(5)

where k_j , j = 1, ..., n - 1 are principal curvatures at the design point that are determined as the eigenvalues of \mathbf{A}' .

$$|\mathbf{A}' - k\mathbf{I}| = 0 \tag{6}$$

Y is also a set of mutually independent, standard, normal random variables obtained by another rotation of **X**, **Y** = **RX**, where **R** is a matrix with the eigenvectors of \mathbf{A}' as column vectors.

Almost all of the SORM formulas use the performance function of (4) to evaluate failure probability. Appendix I lists four of these formulas that will be investigated in this paper.

The currently used SORM approximation exhibits the following problems, which may be considered to be the main reasons SORM has not been used extensively up to now.

- 1. The difference in the principal curvatures between (2) and (5), and the inaccuracy resulting from replacing (2) by (5) have not been adequately investigated. Simpler approximations may be possible.
- 2. The principal curvatures k_j defined by (6) are used in almost all of the previous SORM approximations. In order to obtain the principal curvatures k_j in (4), two matrix rotations, i.e., the rotation to obtain **A** in (2) and the eigenvalue analysis in (6), are needed. These rotations are quite complicated and the definition of the principal curvatures is not easily understood by engineers because the definition is different from that in differential geometry, as investigated in Appendix II in the case of three dimensions.
- 3. Most importantly, the accuracy of these approximations is questionable. The ensuing sections of this paper will show that, although they are quite complicated, almost all of the existing SORM formulas give good approximations for only a small range of curvatures, number of random variables, and first-order reliability indices.

SIMPLE APPROXIMATION OF SECOND-ORDER RELIABILITY METHOD

In order to avoid the difficulties described above, consider the second-order Taylor expansion in standardized space, (2). Forming a plane surface in the $x_n - x_j$ plane though the design point **X***, the intersection curve between the limit state surface (2) and the plane can be expressed as

$$a_{jj}x_j^2 + 2a_{nj}x_j(x_n - \beta_F) + a_{nn}(x_n - \beta_F)^2 - 2(x_n - \beta_F) = 0 \quad (7)$$

The curvature of this curve at the design point is obtained as

$$k'_{j} = a_{jj}, j = 1, \dots, n - 1$$
 (8)

in which a_{ij} , j = 1, ..., n - 1 are diagonal elements of **A**.

The sum of the principal curvatures k_j , j = 1, ..., n - 1 of the limit state surface at the design point can be expressed as the following equation according to differential geometry (Kobayasi 1977):

$$K_{s} = \sum_{j=1}^{n-1} k_{j} = \sum_{j=1}^{n-1} k_{j}' = \sum_{j=1}^{n} a_{jj} - a_{nn}$$
(9)

Because \mathbf{A} is transformed from \mathbf{B} using orthonormal transformation, (10) holds true according to linear algorithms.

$$\sum_{j=1}^{n} a_{jj} = \sum_{j=1}^{n} b_{jj} \quad a_{nn} = \boldsymbol{\alpha}^{T} \mathbf{B} \boldsymbol{\alpha}$$
(10)

where b_{jj} , j = 1, ..., n - 1 are the diagonal elements of **B**. Introducing (10) into (9), K_s can be expressed as

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$$K_s = \sum_{j=1}^n b_{jj} - \boldsymbol{\alpha}^T \mathbf{B} \boldsymbol{\alpha}$$
(11)

Approximating the limit state surface by a rotational parabolic surface of diameter 2R, where R is the average principal curvature radius expressed as

$$R = \frac{n-1}{K_s} \tag{12}$$

the performance functions in standardized space can be expressed simply as (13), which is a special form of (4)

$$G(\mathbf{U}) = -(u_n - \beta_F) + \frac{1}{2R} \sum_{j=1}^{n-1} u_j^2$$
(13)

The limit state surface of (13) is convex to the origin when R is positive and is concave to the origin when R is negative. The performance function $Z = G(\mathbf{U})$ is a combination of a standardized, normal random variable and a random variable of a central chi-square distribution with n - 1 degrees of freedom (Fiessler et al. 1979). The probability $P_f = Prob \{G(\mathbf{U}) < 0\}$ is computed using

$$P_F = \int_0^\infty \Phi\left(\frac{t}{2R} - \beta_F\right) f_{x_{n-1}^2}(t) dt \tag{14}$$

where

$$f_{x_{n-1}^2}(t) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{(n-1)/2}} t^{(n-3)/2} \exp\left(-\frac{t}{2}\right)$$
(15)

Although (13) is considered to be only a special case of (4), the corresponding failure probability can be obtained accurately, and the present study will show that the approximation has an accuracy comparable to other existing formulas. The present approximation has the following characteristics:

- Although the orthogonal transformation matrix **A** is used to deduce the formula of K_s , finally K_s is actually expressed in terms of elements of **B** rather than those of **A**. Therefore, the rotational matrix transformation and eigenvalue analysis are not required for the calculation of K_s .
- The curvature definition used in (11) is the common, easily understood definition, whereas the definition in (6) is different, as shown in Appendix II.
- The computation required for the failure probability (14) does not possess any singularities except at R = 0.
- Both (5) and (13) are approximations of (2), but (13) is much simpler than (5), and the failure probability corresponding to (13) is easier to be obtained.

EMPIRICAL SECOND-ORDER RELIABILITY INDEX

Since a closed form expression for (14) cannot be obtained, numerical integration is necessary. Easy application requires the probability to be presented in closed form. Because it is very difficult to obtain a closed-form result with a high enough degree of accuracy, this study obtained an empirical closedform solution.

According to the variations in the second-order reliability index investigated using four special second-order performance functions (Zhao and Ono 1997), an empirical secondorder reliability index was obtained as follows:

$$\beta_{s} = \left(1 - \frac{K_{s}}{3\beta_{F} + 3(n-1)/K_{s} + 1}\right)\beta_{F} + \frac{1}{2}K_{s} \quad K_{s} \ge 0 \quad (16a)$$
$$\beta_{s} = \left(1 - \frac{K_{s}^{2}}{3(n-\beta_{F} + 3)}\right)\beta_{F} + \frac{1}{2}K_{s} \quad K_{s} < 0 \quad (16b)$$

where K_s = sum of principal curvatures of limit state surface described in (11); R = average principal curvature radius described in (12); n = number of random variables; β_F = firstorder reliability index; β_s = second-order reliability index.

Eq. (16) is very simple and it has been shown in Zhao and Ono (1997) that it has sufficient accuracy in almost all cases that do not have a large number of random variables or small curvature radii. To improve the computational accuracy in the two cases mentioned above, the following empirical secondorder reliability index is obtained from a large number of computations and regressions.

$$\beta_{s} = -\Phi^{-1} \left[\Phi(-\beta_{F}) \left(1 + \frac{\phi(\beta_{F})}{R\Phi(-\beta_{F})} \right)^{-[(n-1)/2](1 + [2K_{s}/10(1+2\beta_{F})])} \right] \\ K_{s} \ge 0$$
(17*a*)

$$\beta_{s} = \left(1 + \frac{2.5K_{s}}{2n - 5R + 25(23 - 5\beta_{F})/R^{2}}\right)\beta_{F} + \frac{1}{2}K_{s}\left(1 + \frac{K_{s}}{40}\right) \quad K_{s} < 0$$
(17b)

Eqs. (17*a*) and (17*b*) have different analytical expressions, they are obtained only by regressions using trial-and-error method. The expression of (17*a*) is similar to that of Koyluoglu's formula (Appendix I) when $k_j = 1/R$ and $k_j > 0$ for $j = 1, \ldots n - 1$, so it can be also interpreted as a correction for Koyluoglu's formula in this case.

Figs. 1–3 depict the differences between the β_s obtained by (16) and (17). Fig. 1 shows the relationship between β_s and



FIG. 1. Variation of Empirical Reliability Index with Respect to Curvature Radius



FIG. 2. Variation of Empirical Reliability Index with Respect to Number of Random Variables



FIG. 3. Variation of Empirical Reliability Index with Respect to First-Order Reliability Index

curvature radius *R* for the number of random variables n = 8, where the three horizontal lines represent three levels of firstorder reliability index $\beta_F = 2$, 3, 4. Fig. 2 shows the relationship between β_s and *n* for R = 5 and R = -5, respectively, where the three horizontal lines represent three levels $\beta_F = 2$, 3, 4, as in Fig. 1. Fig. 3 shows the relationship between β_s and β_F for R = 5 and R = -5, respectively, where two cases of n = 8 and n = 24 are given in the figure.

From the results shown in Figs. 1-3, one can see that both (16) and (17) give good approximations of the exact results for relative large curvature radii and small number of random variables. Because (17) provides better accuracy in the case of small curvature radii and large number of random variables, (17) is proposed as the empirical second-order reliability index in this study.

EXAMPLES AND INVESTIGATIONS

Investigation for Rotational Parabolic Surface

The first example considers the performance function (13) directly. Because (13) is a special form of (4), all the closed form results that are claimed to be accurate for (4) should also be accurate for (13).

The variations of the computational results of second-order reliability index with respect to curvature radius R are shown in Fig. 4 (exact results, those obtained by (17), and those obtained by the four kinds of SORM formulas listed in Appendix I). These are denoted as exact, present, Breitung, Koyluoglu, Cai, and Tvedt, respectively. The exact results are obtained



FIG. 4. Variation of Reliability Index with Curvature Radius for Example 1

using the direct integration of (14). The first-order reliability index is taken to be 2.0, which is depicted as a horizontal line in the figure. The number of random variables is taken to be 8. From Fig. 4, one can see that the second-order reliability indices obtained from all of the formulas improve the firstorder reliability index to some degree. When the absolute curvature radius is large, all of the second-order reliability indices converge, and are close to the exact value as well as the firstorder reliability index. For positive curvature, Tvedt's formula, Koyluoglu's formula, and the present formula give good approximations of the exact results, whereas Cai's formula produces a large error for relatively small radius. For negative curvature, Cai's formula and the present formula give good approximations of the exact results, whereas Tvedt's formula and Koyluoglu's formula produce a large error for relatively small radius (absolute value).

The variations of the computed second-order reliability indices with respect to the number of random variables are shown in Fig. 5, in which the first-order reliability index is taken to be 2.0 and is depicted as a horizontal line in the figure. The curves above the first-order reliability index show the results of the limit state surface convex to the origin (R = 10)and those below it show the results of the limit state surface concave to the origin (R = -10). From Fig. 5, one can find that (1) the larger the number of random variables, the greater the difference between the second-order and first-order reliability indices, and the exact second-order reliability index is approximately proportional to the number of random variables; and (2) all the second-order reliability indices improve the first-order reliability index to some degree. For small number of random variables, all the second-order reliability indices give good approximations of the exact results. When the number of random variables is large, for positive curvature, the present formula gives very good approximations, and Koyluoglu's formula and Breitung's formula give relatively good approximations for the exact results, while Tvedt's formula and Cai's formula produce large errors. When the number of random variables is larger than 16 for Cai's formula and 30 for Tvedt's formula, suitable results cannot be obtained because the failure probability computed by the two formulas is beyond definition. For negative curvature, Cai's formula and the present formula give very good approximations of the exact results, whereas Tvedt's formula, Breitung's formula, and Koyluoglu's formula produce large errors for a relatively large number of random variables.

The variations of the second-order reliability indices with respect to the first-order reliability index are shown in Fig. 6, in which the number of random variables is taken to be 10. The curves above the first-order reliability index show the re-



FIG. 5. Variation of Reliability Index with Number of Random Variables for Example 1



FIG. 6. Variation of Reliability Index with First-Order Reliability Index for Example 1

sults of the limit state surface convex to the origin (R = 5), and those below it show the results of the limit state surface concave to the origin (R = -5). From Fig. 6, one can see that (1) the exact second-order reliability indices are approximately proportional to the first-order reliability index, and all of the second-order reliability indices obtained from each of the formulas improve the first-order reliability index to some degree; and (2) the present formula gives very good approximations for the entire range of the first-order reliability indices for not only positive curvature but also negative curvature. For positive curvature. Breitung's formula and Tvedt's formula provide good results with the increase of the first-order reliability index, whereas Cai's formula provides good results for small first-order reliability indices but cannot provide appropriate results for large first-order reliability indices. For negative curvature, all of the formulas except the present formula produce large errors; Breitung's formula and Tvedt's formula cannot provide appropriate results.

Investigation for General Parabolic Surface

The second example considers the following performance function in standardized space, which has been used as the fourth example by Der Kiureghian (1987). This function is also a special form of (4) but is more general than (13).

$$G(\mathbf{U}) = \beta_F - u_n + \frac{1}{2} \sum_{j=1}^{n-1} j \ a \ u_j^2$$
(18)

The sum of the principal curvatures of the limit state surface for (18) can be readily obtained as $K_s = an(n - 1)/2$ and the corresponding average curvature radius is obtained as R = 2/(na). The exact results used in the following investigations are obtained by the inverse fast Fourier transformation (IFFT) method (Zhao and Ono 1999).

Fig. 7 shows the relationship between curvature radius R and the second-order reliability indices obtained using the present formula, as well as the four formulas listed in Appendix I, with comparison of the exact results, in which the first-order reliability index is taken to be 2.0 and the number of random variables is taken to be 8. From Fig. 7, one can see that there is good agreement between the present formula and Tvedt's formula, both give the best approximations of the exact results for a relatively large R (larger than 5) in this case. Breitung's and Cai's formulas have significant errors when the curvature radius is smaller than 10.

The variations of the second-order reliability index with respect to the number of random variables are shown in Fig. 8, in which the first-order reliability index is taken to be 2.0, and the curvature radius is taken to be 5. From Fig. 8, one can see



FIG. 7. Relationship between Reliability Index and Curvature Radius for Example 2



FIG. 8. Relationship between Reliability Index and Number of Random Variables for Example 2

that only the proposed formula gives a good approximation of the correct results when the number of random variables is relatively large. Tvedt's formula and Cai's formula produce significant errors for a large number of random variables in this case.

The variations of the second-order reliability indices with respect to the first-order reliability index are shown in Fig. 9, where the number of random variables is taken to be 8 and the curvature radius is taken to be 5. One can see that all of the second-order reliability indices give good approximations



FIG. 9. Relationship between Reliability Index and First-Order Reliability Index for Example 2

of the exact results for a large first-order reliability index. Cai's formula and Breitung's formula produce significant errors for a small first-order reliability index in this case.

Investigation for Spherical Surface

The third example considers the following performance function in standardized space, which is the general case of the practical examples used by Cai (1994) and Koyluoglu (1994).

$$G(\mathbf{U}) = R^2 - \sum_{j=1}^{n} (u_j - \lambda_j)^2$$
(19)

The limit state surface of (19) is a hypersphere concave to the origin with radius *R* and center at point $(\lambda_j, j = 1, ..., n)$. $y = G(\mathbf{U})$ is a random variable having the noncentral chisquared distribution, and the exact value of probability $P_F =$ $Prob\{G(\mathbf{U}) < 0\}$ is computed directly using this distribution (Sankaran 1959, 1963). The distance from origin to the spherical center δ is expressed as

$$\delta^2 = \sum_{i=1}^n \lambda_i^2 = (R - \beta_F)^2$$
⁽²⁰⁾

The design point \mathbf{U}^* and the directional vector α at U^* are expressed as

$$\mathbf{U}^* = \left\{-\frac{\beta_F}{\delta}\lambda_j, j = 1, \dots, n\right\}; \quad \alpha = \left\{-\frac{\lambda_j}{\delta}, j = 1, \dots, n\right\}$$
(21)

from which the scaled Hessian matrix is obtained as

$$\mathbf{B} = \frac{1}{R} \mathbf{I} \tag{22}$$

When using previous SORM formulas, a rotation matrix **H** should be established to obtain **A** in (2); **A** is then simplified to **A**' and the principal curvatures are obtained from the eigenvalue analysis of **A**' using (6). The results are obtained as $k_1 = k_2 = \cdots = k_{n-1} = 1/R$. When using the empirical reliability index (17), the sum of the principal curvatures can be readily obtained as $k_s = (n - 1)/R$ without matrix rotation and eigenvalue analysis.

The variations of the second-order reliability indices with respect to the number of random variables are shown in Fig. 10, in which the first-order reliability index is taken to be 3.0, and curvature radius is taken to be -10. From Fig. 10, one can see that for small number of random variables, all of the formulas give good approximations of the exact results, but only the present formula gives good approximations when the



FIG. 10. Relationship between Reliability Index and Number of Random Variables for Example 3



FIG. 11. Results of Reliability Index for Example 4

number of random variables is relatively large. Both Koyluoglu's and Cai's formulas are claimed to be accurate, because the writers investigated the formulas only for the case of n =3 (Koyluoglu and Nielsen 1994; Cai and Elishakoff 1994).

Investigation for Paraboloid with Unevenly Distributed Curvatures

The fourth example considers the following performance function in standardized space:

$$G(\mathbf{U}) = \beta_F - u_8 + \frac{1}{2} \sum_{j=1}^{\prime} a^j u_j^2$$
(23)

where a = a factor with value from -1 to 1. Because a^{j} changes for different *j*, the paraboloid expressed by (23) has unevenly distributed curvatures. The smaller the absolute value of *a*, the more unevenly the curvature is distributed.

The variations of the second-order reliability indices with respect to a are shown in Fig. 11, in which the first-order reliability index is taken to be 2.0. The exact results are obtained by IFFT method (Zhao and Ono 1999). From Fig. 11, one can see that for a > 0, which implies that the curvatures have the same signs, the present formula gives a better approximation compared to other formulas. For a < 0, which implies that the curvatures have different signs, all of the formulas including the present formula produce significant errors.

CONCLUSIONS

- 1. For practical application of the second-order reliability method, a simple approximation has been suggested and an empirical second-order reliability index proposed.
- 2. The second-order reliability index is approximately proportional to the number of random variables and the firstorder reliability index.
- 3. All of the four formulas of SORM investigated in this paper—Breitung's formula, Tvedt's formula, Koyluoglu's formula, and Cai's formula—work well for the case of a large curvature radius or a small number of random variables.
- 4. The empirical second-order reliability index gives the best approximation of the exact results for a large range of curvature radii, number of random variables, and first-order reliability index. Furthermore, it can be easily calculated with little additional effort after FORM, without having to compute the rotational transformation of the Hessian matrix or perform eigenvalue analysis.
- 5. For limit state surfaces having curvatures of different signs, all of the formulas including the present one produce significant errors.

(4).

Breitung's Formula

Breitung (1984) has derived an asymptotic formula of the failure probability which asymptotically approaches the exact results as $\beta_F \rightarrow \infty$ with $\beta_F k_i$ fixed.

APPENDIX I. PREVIOUS FORMULAS OF SORM

The investigations in this paper used the following four

kinds of SORM formulas, denoted as Breitung's, Koyluoglu's,

Cai's, and Tvedt's. These four formulas are derived to ap-

proximate the failure probability for the performance function,

$$P_{f} = \Phi(-\beta_{F}) \prod_{j=1}^{n-1} (1 + \beta_{F} k_{j})^{-1/2}$$
(24)

Tvedt's Formula

For moderate β_F , Tvedt (1983) introduced a three-term approximation in which the last two terms can be interpreted as the correction for Breitung's formula.

$$P_{f} = \Phi(-\beta_{F}) \prod_{j=1}^{n-1} (1 + \beta_{F}k_{j})^{-1/2} + A_{2} + A_{3}$$
(25*a*)

 $A_2 = [\beta_F \Phi(-\beta_F) - \phi(-\beta_F)]$

$$\cdot \left\{ \prod_{j=1}^{n-1} \left(1 + \beta_F k_j \right)^{-1/2} - \prod_{j=1}^{n-1} \left(1 + (\beta_F + 1)k_j \right)^{-1/2} \right\}$$
(25b)

$$A_{3} = (\beta_{F} + 1)[\beta_{F}\Phi(-\beta_{F}) - \phi(-\beta_{F})]$$

$$\cdot \left\{ \prod_{j=1}^{n-1} (1 + \beta_{F}k_{j})^{-1/2} - \operatorname{Re}\left[\prod_{j=1}^{n-1} (1 + (\beta_{F} + i)k_{j})^{-1/2} \right] \right\}$$
(25c)

Koyluoglu's Formula

Koyluoglu and Nielsen (1994) proposed three formulas, of which the one-term approximation is the simplest and is claimed to also have good accuracy.

$$P_{f} = \Phi(-\beta_{F}) \prod_{j=1}^{n-1} \left(1 + k_{j} \frac{\phi(\beta_{F})}{\Phi(-\beta_{F})} \right)^{-1/2}$$

for $k_{j} > 0, j = 1, 2, ..., n - 1$ (26)

$$P_{f} = 1 - \Phi(\beta_{F}) \prod_{j=1}^{n-1} \left(1 - k_{j} \frac{\phi(\beta_{F})}{\Phi(\beta_{F})} \right)^{-1/2}$$

for $k_{j} < 0, j = 1, 2, ..., n - 1$ (27)

Cai's Formula

Cai and Elishakoff (1994) introduce a series formula. The three-term approximation is suggested for practical purposes.

$$P_{f} = \Phi(-\beta_{F}) - \phi(\beta_{F})(D_{1} + D_{2} + D_{3})$$
(28*a*)

$$D_1 = \sum_{j=1}^{n-1} \lambda_j \tag{28b}$$

$$D_2 = -\frac{1}{2} \beta_F \left(3 \sum_{j}^{n-1} \lambda_j^2 + \sum_{j \neq l}^{n-1} \lambda_j \lambda_l \right)$$
(28c)

$$D_3 = \frac{1}{6} \left(\beta_F^2 - 1\right) \left(15 \sum_{j=1}^{n-1} \lambda_j^3 + 9 \sum_{j\neq l}^{n-1} \lambda_j^2 \lambda_l + \sum_{j\neq l\neq m}^{n-1} \lambda_j \lambda_l \lambda_m\right) \quad (28d)$$

where $\lambda_j = 1/2k_j, j = 1, 2, ..., n - 1$

APPENDIX II. INVESTIGATION OF DEFINITION OF CURVATURE IN SORM

The investigation is conducted only for the case of three dimensions. Function (5) can be expressed as (29) when $G'(\mathbf{X}) = 0$, which represents the limit state surface.

$$x_n = f(\mathbf{X}') = \beta + \frac{1}{2} \{\mathbf{X}'\}^T [\mathbf{A}'] \{\mathbf{X}'\}$$
(29)

where $\mathbf{X}' = \{x_1, x_2\}, x_n = x_3$.

According to the definition of principal curvature in differential geometry, the principal curvature is given by

$$\frac{1}{N} \left[\mathbf{A}' \right] - k \begin{bmatrix} 1 + \left(\frac{\partial f}{\partial x_1} \right)^2 & \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} & 1 + \left(\frac{\partial f}{\partial x_2} \right)^2 \end{bmatrix} = 0 \quad (30)$$

where

$$N = \sqrt{1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)}$$

This clearly differs from (6).

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APPENDIX III. REFERENCES

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APPENDIX IV. NOTATION

The following symbols are used in this paper:

- \mathbf{A} = scaled second-order derivatives of $G(\mathbf{U})$ at \mathbf{U}^* in rotated space;
- \mathbf{B} = scaled second-order derivatives of $G(\mathbf{U})$ at \mathbf{U}^* ;
- b_{ii} = diagonal element of **B**;
- G = performance function;
- **H** = rotation matrix;
- \mathbf{I} = unit diagonal matrix;
- k_j = principal curvature at U*;
- K_s = sum of principal curvatures of limit state surface;
- n = number of random variables;
- P_f = failure probability;
- \mathbf{R} = rotation matrix;
- R =curvature radius;
- U = standard, normal random variables;
- \mathbf{U}^* = design point in *u*-space;
- \mathbf{X} = standard, normal random variables in rotated space;
- \mathbf{Y} = standard, normal random variables in rotated space;
- α = directional vector at design point in *u*-space;
- β_F = first-order reliability index;
- β_s = second-order reliability index;
- $\Phi(x) = \text{standard, normal probability distribution with argument}$
- $\phi(x)$ = standard, normal density distribution with argument *x*; and
- ∇G = gradient of G at U*.