Second-Order Third-Moment Reliability Method

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Abstract: In the second-order reliability method (SORM), the failure probability is generally estimated using a parabolic approximation of a performance function. In the present paper, the moment properties of a second-order approximation of performance functions are investigated, and a moment approximation for a second-order reliability method and a simple second-order third-moment reliability index are proposed for the estimation of failure probability corresponding to both the simple and general parabolic approximations. Based on the property that the parabolic approximation approaches a unit normal random variable, the ranges of three parameters are investigated: the number of variables, the principal curvature, and the first-order reliability index. A simple analytical judgment formula is derived, which can help us judge when the first-order reliability method is sufficiently accurate and when the SORM is required. A simple second-order second-moment reliability index is also proposed for problems with relatively small principal curvatures. Through some numerical examples, the simplicity and accuracy of the second-order second- and third-moment reliability indices are demonstrated.

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Introduction

A fundamental problem encountered in structural reliability theory is the computation of failure probability described as a multifold probability integral. The difficulty in computing this probability has led to the development of various approximation methods (Ang and Tang 1984). Of interest here is the second-order reliability method (SORM), which is usually used to improve the accuracy of the first-order reliability method (FORM) when the performance function has strong nonlinearity and the first-order approximation is not sufficiently accurate. In the SORM, the limit state surface is approximated by a second-order surface at the design point in transformed standard normal space, and the failure probability is usually estimated using a parabolic surface. Two such approximations have been developed. The first is the general parabolic approximation, which is expressed as (Breitung 1984)

$$G_S(\mathbf{U})=\beta_F-u_n+\frac{1}{2}\sum_{j=1}^{n-1} k_j j_j^2 \tag{1}$$

where u_j, $j=1,\ldots,n$, are standard normal random variables and k_j, $j=1,\ldots,n-1$, are principal curvatures that are determined as the eigenvalues of a matrix having $n-1$ columns and $n-1$ rows approximately transformed from the scaled Hessian matrix (Breitung 1984) at the design point obtained by the FORM. β_F is the first-order reliability index.

The second approximation is the simple parabolic approximation, which is expressed as

$$G_S(\mathbf{U})=\beta_F-u_n+\frac{1}{2R}\sum_{j=1}^{n-1} u_j^2 \tag{2}$$

(Zhao and Ono 1999b) where R=average principal curvature radius which can be obtained without rotational matrix transformation or eigenvalue analysis of the scaled Hessian matrix. n is the number of random variables.

For the general parabolic approximation, numerous studies have contributed to the development of approximations of closed form. Breitung (1984) derived an asymptotic formula that approaches the exact failure probability as $\beta_F\to\infty$ in which $\beta_F k_i$ is fixed. A Taylor series expansion in closed form was derived by Cai and Elishakoff (1994). The formula can be interpreted as a moment formula using the first few moments of G_S about β_F.

For the simple parabolic approximation, an empirical second-order reliability index corresponding to Eq. (2) was given by Zhao and Ono (1999b,c). although the applicable ranges of the three parameters, i.e., n, R, and β_F, for the empirical second-order reliability index are larger than those in other SORM formulas of closed form, the index is an empirical formula and has entirely different forms for positive and negative curvature radii.

In the present paper, the moment properties of the second-order approximation of the performance function are investigated. Using the fact that the parabolic approximation approaches a normal random variable, a moment approximation of second-order reliability and simple second-order, second- and third-moment reliability indices are proposed.

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Second-Order Third-Moment Reliability Index for Simple Parabolic Approximation

In order to derive the second-order third-moment method, we first investigate the moment properties of the simple parabolic performance function. Using the definition of the probability moment, the first three central moments for G_S expressed in Eq. (2) are obtained as

$$\mu_S=\beta_F+\frac{n-1}{2R},\quad \sigma_S^2=1+\frac{n-1}{2R^2},\quad \alpha_{3S}\,\sigma_S^3=\frac{n-1}{R^3} \tag{3}$$

where μ_S and σ_S=mean value and standard deviation of the second-order performance function $G_S(\mathbf{U})$, respectively; and α_{3S}=third dimensionless central moment.

From Eq. (3), it can be seen that α_{3S} rapidly approaches 0 as R increases. Because $\alpha_{3S}=0$ is a moment property of a normal random variable, this implies that the simple parabolic approximation rapidly converges to a normal random variable as R increases. In order to confirm this assumption, the rth cumulant K_r of Eq. (2) is obtained as

$$K_r=\frac{(r-1)!(n-1)}{2R^r}\quad \text{for } r>2 \tag{4}$$

(see the Appendix). Eq. (4) shows that K_r rapidly approaches 0 with increase in R. In other words, G_S satisfies the prerequisite of the Cornish-Fisher expansion (Stuart and Ord 1987).

A standardized random variable $x_s=(G_S-\mu_S)/\sigma_S$ can be approximately transformed to a standard normal random variable u, using the following first polynomial of the inverse Cornish-Fisher expansion (Stuart and Ord 1987):

$$u=x_s-\frac{1}{6}\alpha_{3S}(x_s^2-1) \tag{5}$$

Since

$$\text{Prob}[G_S\leqslant 0]=\text{Prob}\left[x_s\leqslant-\frac{\mu_S}{\sigma_S}\right]$$
$$=\text{Prob}[x_s\leqslant-\beta_{\text{SOSM}}] \tag{6}$$

the reliability index corresponding to the simple parabolic approximation is obtained as

$$\beta_{\text{SOTM}}=\beta_{\text{SOSM}}+\frac{1}{6}\alpha_{3S}(\beta_{\text{SOSM}}^2-1) \tag{7}$$

where

$$\beta_{\text{SOSM}}=\frac{\mu_S}{\sigma_S}=\frac{\beta_F+(n-1)/2R}{\sqrt{1+(n-1)/2R^2}} \tag{8}$$

Since the third and second moments are used in Eqs. (7) and (8), respectively, they are referred to as the second-order third-moment (SOTM) reliability index and the second-order second-moment (SOSM) reliability index, respectively. In particular, when $\alpha_{3S}=0$, Eq. (7) degenerates to $\beta_{\text{SOTM}}=\beta_{\text{SOSM}}$. When the curvature radius is sufficiently large, σ_S approaches 1 and α_{3S} will approach 0, and then the SOTM and SOSM reliability indices degenerate to $\beta_{\text{SOTM}}=\beta_{\text{SOSM}}=\beta_F$.

Second-Order Third-Moment Reliability Index for General Parabolic Approximation

For the general parabolic approximation, the rth cumulant K_r is given as

$$K_r=\frac{(r-1)!}{2}\sum_{j=1}^{n-1}k_j^r\quad \text{for } r>2 \tag{9}$$

(see the Appendix). Eq. (9) shows that K_r rapidly approaches 0 with decreasing $\Sigma_{j=1}^{n-1}k_j^r$. This implies that the general parabolic approximation rapidly converges to a normal random variable as $\Sigma_{j=1}^{n-1}k_j^r$ decreases. In other words, the general parabolic approximation also satisfies the prerequisite of the Cornish-Fisher expansion. Substituting the first three moments of the general parabolic approximation into Eq. (7), the SOTM reliability index for the general parabolic approximation can easily be obtained. The first three central moments of Eq. (1) are given by

$$\mu_S=\beta_F+\frac{1}{2}\sum_{i=1}^{n-1}k_i,\quad \sigma_S^2=1+\frac{1}{2}\sum_{i=1}^{n-1}k_i^2$$
$$\alpha_{3S}\sigma_S^3=\sum_{i=1}^{n-1}k_i^3 \tag{10}$$

The SOSM reliability index for the general parabolic approximation is simply expressed as

$$\beta_{\text{SOSM}}=\frac{\mu_S}{\sigma_S}=\frac{\beta_F+\frac{1}{2}\sum_{i=1}^{n-1}k_i}{\sqrt{1+\frac{1}{2}\sum_{i=1}^{n-1}k_i^2}} \tag{11}$$

Applicable Range of First-Order Reliability Method

The problem of the accuracy of the FORM has been examined by many studies through a large number of examples (e.g., Der Kiureghian and De Stefano 1991). However, detailed reports of the parameter ranges for which it is sufficiently accurate are rare. An empirical investigation was conducted but under only one level of accuracy (Zhao and Ono 1999a). Using the SOSM and SOTM reliability indices presented in this paper, the applicable range of the FORM can be obtained analytically.

Since the FORM is accurate only in cases where the curvature radius is very large, in the range of R where the accuracy of the FORM was investigated, the SOSM reliability index is sufficiently accurate. As shown in Eq. (3), the standard deviation σ_S and the third dimensionless central moment α_{3S} approach 1 and 0, respectively, as the curvature radius R increases, and therefore the second-order reliability index β_S used in the investigation of the applicable range for FORM can be expressed as

$$\beta_S=\beta_F+\frac{1}{2}K_S=\beta_F+\frac{n-1}{2R} \tag{12}$$

where K_S=total principal curvature of the limit state surface at the design point.

Using $|\beta_S-\beta_F|/\beta_S\leqslant\gamma$ as a criterion for the first-order reliability index to judge whether the FORM is sufficiently accurate, the range of the average curvature radius or the sum of the principal curvature can be given as

$$|R| \geqslant \frac{n-1}{2\gamma\beta_F} \quad \text{or} \quad |K_S| \leqslant 2\gamma\beta_F \tag{13}$$

where γ is the tolerance error of the FORM.

Using Eq. (13), the applicable range for which the FORM is sufficiently accurate can be judged quite conveniently. For example, if γ is taken to be 2%, then $|K_S| \leqslant 0.04\beta_F$ is necessary, and furthermore, for $n=5$, $|R| \geqslant 100/\beta_F$ is also necessary.

Similarly, for the general parabolic approximation, the second-order reliability index β_S used in the investigation of the applicable range for the FORM can be expressed as

$$\beta_S = \beta_F + \frac{1}{2}\sum_{i=1}^{n-1} k_i \tag{14}$$

and the applicable range for the FORM can easily be expressed as

$$|K_S| = \left|\sum_{i=1}^{n-1} k_i\right| \leqslant 2\gamma\beta_F \tag{15}$$

Numerical Examples

Investigation for General Paraboloid with Unevenly Distributed Curvatures

The first example considers the following performance function in standard normal space:

$$G(\mathbf{U}) = \beta_F - u_8 + \frac{1}{2}\sum_{j=1}^{7} a^j u_j^2 \tag{16}$$

where a is a factor having a value from -0.4 to 0.4. Because a^j changes according to j, the paraboloid expressed by Eq. (16) possesses unevenly distributed curvatures. For this example investigated by Zhao and Ono (1999b), none of the currently used SORM formulas, including the empirical second-order reliability index, gave satisfactory results.

Using Eq. (15), the range of K_S for which the FORM is accurate can be readily obtained as $K_S \leqslant 0.08$ for $\beta_F = 2$ and $\gamma = 2\%$. The variations of K_S with respect to a are shown in Fig. 1(a), in which the shadow region indicates K_S for which the error of β_F is less than $\gamma = 2\%$. From Fig. 1(a), it can be seen that the absolute value of K_S increases as the absolute value of a increases, and enters the shadow region for which the error of β_F is less than $\gamma = 2\%$ when $-0.085 < a < 0.075$. At the critical values of $a = -0.085$ and 0.075, the corresponding values $\beta_{SOSM} = 1.9573, 2.0377$ and $\beta_{SOTM} = 1.9570, 2.0379$, respectively, are obtained. Once can see that the β_{SOSM} value is almost equal to β_{SOTM}, and the differences between β_F and β_{SOTM} are about $\gamma = 2\%$. This implies that the assumption used in deriving Eqs. (13) and (15) was appropriate and that Eqs. (13) and (15) can be used to estimate the applicable range of the FORM.

The variations of the SOSM reliability index obtained using Eq. (11) and the SOTM reliability index obtained using Eqs. (7) and (11) are depicted in Fig. 1(b) with a comparison of the empirical second-order reliability index and the exact results obtained by the inverse fast Fourier transformation (IFFT) method (Zhao and Ono 1999c). Fig. 1(b) shows that for $a > 0$, implying that the curvatures have the same signs, both the empirical and the SOTM reliability indices provide good approximations of the exact reliability index. One can also see that the SOSM reliability index produces significant errors with increase of a, since K_S increases with a as shown in Fig. 1(a). For $a < 0$, implying that the curvatures have different signs and that the total principal curvature is negative, the SOTM reliability index provides good approximation of the exact reliability index while the empirical reliability index produces significant errors. Since K_S is quite small for $a < 0$ as shown in Fig. 1(a), the SOSM reliability index also provides good approximation of the exact reliability index. The results of this example imply that the proposed SOTM reliability index can be applied to problems with unevenly distributed principal curvatures.

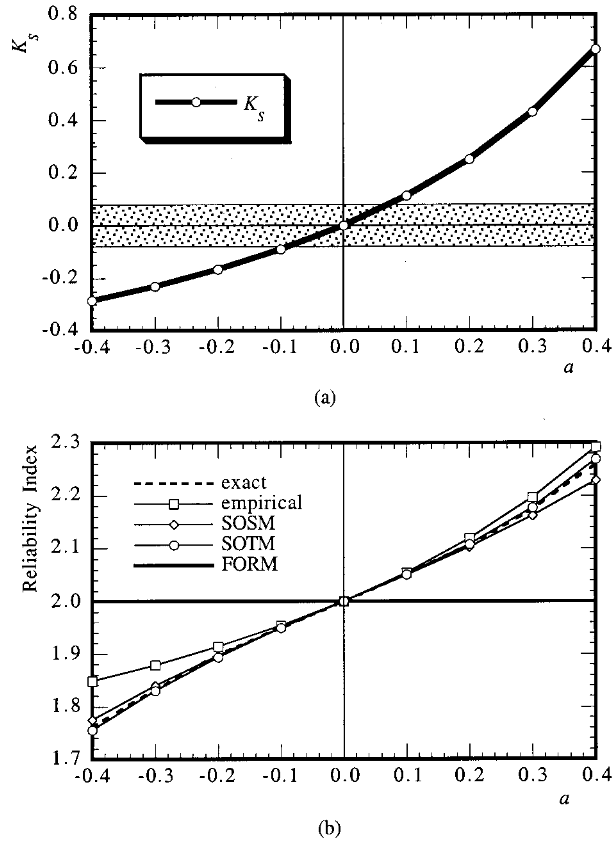

Fig. 1. Variation of reliability index and principal curvature for Example 1

Second-Order Reliability Analysis of Frame Structure

The second example considers a six-story, two-bay frame structure that was introduced as Example 5 by Der Kiureghian and De Stefano (1991), to examine the efficiency of the SORM for a problem with a large number of random variables. The structure is assumed to be elastic, but geometric nonlinearity due to the P-Δ effect is considered. The problem is defined by 99 random variables which include all nodal loads and individual member properties. With the aid of the first six principal curvatures provided by Der Kiureghian and De Stefano (1991) using their point fitting SORM, the general parabolic approximations of the problems with the P-Δ effect are expressed as

$$G_S = 1.971 - u_7 + \tfrac{1}{2}(-0.3038u_1^2 - 0.164u_2^2 - 0.0978u_3^2 - 0.0521u_4^2 - 0.0318u_5^2 + 0.0187u_6^2) \tag{17}$$

where the first term 1.971 in Eq. (17) is the first-order reliability index corresponding to a failure probability of 0.02435.

Using Eq. (17), the total principal curvature can be readily obtained as $K_S = -0.6308$. Using Eq. (15), the range of K_S for which the FORM is adequate can be obtained as $|K_S| \leq 0.0788$ for $\gamma = 2\%$. One can easily understand that the error of the FORM for this problem will be much larger than $\gamma = 2\%$. The second-order probability estimate given by Der Kiureghian and De Stefano (1991) is 0.05508 corresponding to a reliability index of 1.597.

For the general parabolic approximation given in Eq. (17), the first three moments are readily obtained as $\mu_S = 1.656$, $\sigma_S = 1.033$, and $\alpha_3 = -0.0305$, and the SOSM and SOTM reliability indices are 1.603 and 1.595, which correspond to failure probabilities of 0.0544 and 0.0553, respectively. One can see that the SOTM reliability index is in good agreement with the second-order probability estimates of Der Kiureghian and De Stefano (1991). Since α_3 is quite small, the SOSM reliability index also provides a good result.

Using the simple parabolic approximation for Eq. (17), the average curvature radius is -9.551, the first three moments are $\mu_S = 1.656$, $\sigma_S = 1.0164$, and $\alpha_3 = -0.00664$, and the SOSM and SOTM reliability indices are 1.629 and 1.627, which correspond to failure probabilities of 0.0516 and 0.0519, respectively. One can see that both the SOSM and SOTM reliability indices using the simple parabolic approximation provide comparable results with those using the general parabolic approximation.

Conclusions

1. The parabolic approximation of a performance function approaches a normal random variable as the curvature radius increases.
2. The proposed simple SOSM and SOTM reliability indices have the same form for both negative and positive principal curvatures, and are effective for both simple and general parabolic approximations of a performance function.
3. The ranges of the three parameters for which the FORM was sufficiently accurate were investigated and a simple analytical judgment formula was derived for use in determining when the FORM is sufficiently accurate and when the SORM is required.

Appendix: Cumulants of Parabolic Approximations

For the simple parabolic approximation, the characteristic function is given by

$$Q(t) = \exp(i\beta_F t)\exp\left(-\frac{t^2}{2}\right)\left(1 - \frac{it}{R}\right)^{-(n-1)/2} \tag{18}$$

The expansion of $\ln[Q(t)]$ is readily given as

$$\ln[Q(t)] = \beta_F(it) + \frac{1}{2}(it)^2 + \frac{n-1}{2}\sum_{r=1}^{\infty}\frac{(it)^r}{rR^r} \tag{19}$$

Since the rth cumulant K_r is the coefficient of $(it)^r/r!$ in $\ln[Q(t)]$, K_r is easily obtained as Eq. (4).

Similarly, for the general parabolic approximation, the expansion of $\ln[Q(t)]$ is given as

$$\ln[Q(t)] = \beta_F(it) + \frac{1}{2}(it)^2 + \frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{r}\left(\sum_{r=1}^{\infty}k_j^r\right)(it)^r \tag{20}$$

K_r is easily obtained as Eq. (9).

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