Estimating Joint Failure Probability of Series Structural Systems

Yan-Gang Zhao, M.ASCE[1]; Wei-Qiu Zhong[2]; and Alfredo H.-S. Ang, Hon.M.ASCE[3]

Abstract: The failure probability of a series structural system theoretically involves multidimensional integration and is usually difficult to calculate. The search for efficient computational procedures for estimating system reliability has resulted in several approaches, including bounding techniques and efficient Monte Carlo simulations. For the narrow bound method, the joint failure probability of every pair of failure modes needs to be calculated. In the present paper, in order to improve the accuracy of the narrow bound estimation method, a computationally effective point estimation method for calculating the joint probability of every pair of failure modes is proposed and examined for series systems. Based on the computational results of several illustrative examples, it can be seen that the results by the present method are in good agreement with those obtained through integration.

DOI: 10.1061/(ASCE)0733-9399(2007)133:5(588)

CE Database subject headings: Structural reliability; System reliability; Joints; Failures; Integrated systems.

Introduction

The evaluation of system reliability for structures has been an active area of research for over three decades. According to the logical relationship of the failure modes of structures, structural systems can be divided into three types: series systems, parallel systems, and hybrid structural systems. Of interest here is the reliability assessment of series systems, which is encountered most frequently in practical design and analysis.

The failure probability of a series structural system theoretically involves multidimensional integration, which is usually difficult to evaluate, especially for structures of practical significance. The search for efficient computational procedures for estimating system reliability has resulted in several approaches, including bounding techniques and efficient Monte Carlo simulations (MCS).

The bounding methods include the wide bound estimation method and the narrow bound estimation method. For the narrow bound method, the joint failure probability of every pair of failure modes needs to be calculated. In this study, in order to improve the accuracy of the narrow bound estimation method, a point estimation of the joint failure probability of series structural systems is proposed and examined.

[1]Associate Professor, Dept. of Architecture, Nagoya Institute of Technology, Gokiso-cho, Shyowa-ku, Nagoya 466-8555, Japan.

[2]Assistant Professor, Venture Business Laboratory, Nagoya Institute of Technology, Gokiso-cho, Shyowa-ku, Nagoya 466-8555, Japan.

[3]Research Professor, Dept. of Civil and Environmental Engineering, Univ. of California, Irvine, CA 92697-2175.

Note. Associate Editor: Arvid Naess. Discussion open until October 1, 2007. Separate discussions must be submitted for individual papers. To extend the closing date by one month, a written request must be filed with the ASCE Managing Editor. The manuscript for this paper was submitted for review and possible publication on February 21, 2006; approved on October 9, 2006. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 133, No. 5, May 1, 2007. ©ASCE, ISSN 0733-9399/2007/5-588–596/$25.00.

Review of System Reliability of Series Systems

Consider a series structural system with k possible failure modes, and let the performance function for failure mode i be given by

$$g_i(\mathbf{X}) = g_i(x_1, x_2, \cdots, x_n); \quad i = 1, 2, \cdots, k \tag{1}$$

where $x_1, x_2, \ldots, x_n$ are the basic random variables and $g_i(\cdot)$ is the performance function.

Define the failure event for failure mode i as

$$E_i = [g_i(\mathbf{X}) \leqslant 0] \tag{2}$$

Since the occurrence of any failure event E_i will cause the failure of the structure, the failure event E of the structure is the union of all the possible failure modes, which can be expressed as

$$E = E_1 \cup E_2 \cup \cdots \cup E_k \tag{3}$$

In structural reliability theory, the failure probability P_f of a series structural system corresponding to the occurrence of the event E of Eq. (3) theoretically involves the following integration:

$$P_f = \int_{(E_1 \cup E_2 \cup \cdots \cup E_k)} \cdots \int f_{x_1, x_2, \cdots, x_n}(x_1, x_2, \cdots, x_n) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \tag{4}$$

where $f(\cdot)$ is the pertinent joint probability density function.

The evaluation of the above multidimensional integration is often difficult, especially for structures of practical dimensions. For this reason, approximate methods have been proposed and developed. These include the following "wide" bound technique for the failure probability of series structural systems [e.g., Cornell (1967)]:

$$\max_{1 \leqslant i \leqslant k} (P_{fi}) \leqslant P_f \leqslant 1 - \prod_{i=1}^{k} (1 - P_{fi}) \tag{5}$$

where P_{fi} is the failure probability of the ith failure mode.

Since only the failure probability of a single failure mode is considered and the correlation of the failure modes is neglected, the above wide bound estimation method is simple to evaluate; however, the bounds can be very wide, especially for a complex system.

A "narrow" bound estimation method for the failure probability of series systems is also available (Ditlevsen 1979)

$$P_{f1}+\sum_{i=2}^{k}\max\left(P_{fi}-\sum_{j=1}^{i-1}P_{fij},0\right)\leqslant P_f\leqslant\sum_{i=1}^{k}P_{fi}-\sum_{i=2}^{k}\max_{j<i}(P_{fij}) \tag{6}$$

where P_{fij} is the joint probability of the simultaneous occurrences of the ith and jth failure modes. The left- and right-hand sides of Eq. (6) are, respectively, the lower bound and upper bound of the failure probability of a series structural system with k potential failure modes. Observe that because the joint probability of simultaneous failures of every pair of failure modes must be evaluated, the resulting bounds of Eq. (6) are narrower than those of Eq. (5).

As is well known, P_{fij} can be expressed by (Ang and Tang 1984)

$$P_{fij}=\Phi_2(-\beta_i,-\beta_j,\rho_{ij})=\int_{-\infty}^{-\beta_i}\int_{-\infty}^{-\beta_j}\phi_2(x_i,x_j,\rho_{ij})\mathrm{d}x_i\mathrm{d}x_j \tag{7}$$

where

$$\phi_2(x_i,x_j,\rho_{ij})=\frac{1}{2\pi\sqrt{1-\rho_{ij}^2}}\exp\left(-\frac{1}{2}\cdot\frac{x_i^2+x_j^2-2\rho_{ij}x_ix_j}{1-\rho_{ij}^2}\right) \tag{8}$$

The reliability indices β_i and β_j correspond to the ith and jth failure modes, respectively; ρ_{ij} is the correlation coefficient between the ith and jth failure modes; and $\phi_2(\cdot)$ and $\Phi_2(\cdot)$ are the probability density function and cumulative distribution function, respectively, of 2D standard normal distribution.

Eq. (7) is an accurate expression for P_{fij}. To obtain the results, however, numerical integrations would be needed. To avoid such numerical integrations, further approximations are often adopted (involving further bounds for P_{fij}). Specific formulas for evaluating the lower and upper bounds of the joint failure probability P_{fij} were proposed by Ditlevsen (1979) as follows:

$$\left.\begin{array}{ll}\max[\mathrm{P(A)},\mathrm{P(B)}]\leqslant P_{fij}\leqslant\mathrm{P(A)}+\mathrm{P(B)} & \rho_{ij}\geqslant 0\\ 0\leqslant P_{fij}\leqslant\min[\mathrm{P(A)},\mathrm{P(B)}] & \rho_{ij}<0\end{array}\right\} \tag{9}$$

where

$$\mathrm{P(A)}=\Phi(-\beta_i)\Phi\left(-\frac{\beta_j-\rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}}\right)$$
$$\mathrm{P(B)}=\Phi(-\beta_j)\Phi\left(-\frac{\beta_i-\rho_{ij}\beta_j}{\sqrt{1-\rho_{ij}^2}}\right) \tag{10}$$

Since Eq. (9) is a bound rather than a specific value, it is not convenient to use in Eq. (6). Feng (1989) gave a point estimate for the joint failure probability P_{fij} as

$$P_{fij}=[\mathrm{P(A)}+\mathrm{P(B)}][1-\arccos(\rho_{ij})/\pi] \tag{11}$$

where the definitions of $P(A)$ and $P(B)$ are the same as those in Eq. (9) and can also be calculated by Eq. (10). Since Eq. (11) is a specific value rather than a bound, it is convenient and considered to have high accuracy to be used in Eq. (6) for obtaining the narrow bounds of the system reliability (Wu and Burnside 1990; Song 1992; Penmesta and Grandhi 2002; Adduri et al. 2004). As described by Feng (1989), when the correlation coefficient $\rho_{ij}=0$ or 1, Eq. (11) gives accurate solutions, whereas when $0<\rho_{ij}<1$, the calculational accuracy is reasonably high, especially when $\rho_{ij}\leqslant 0.6$. However, as will be shown later, the lower bound

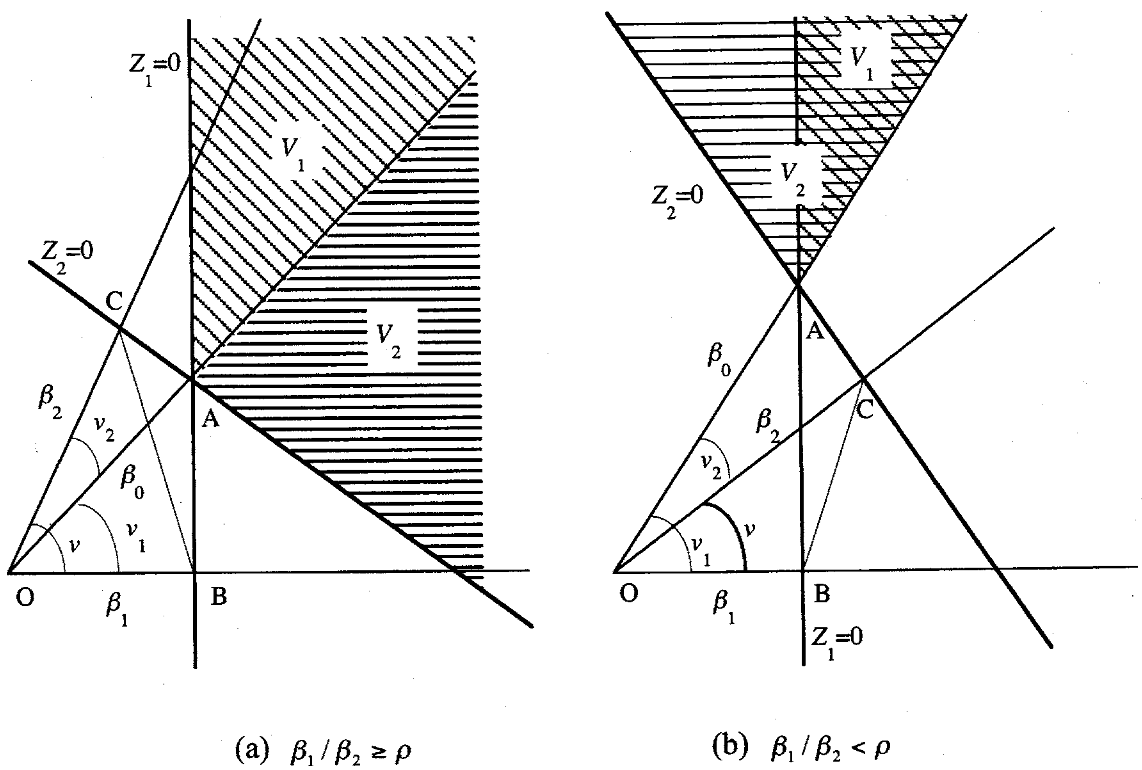


Fig. 1. Probability calculation of P_{f12} in standard normal space

obtained with Eq. (11) can sometimes be lower than the lower bound given by Eq. (9).

The present paper proposes an alternate method for estimating the joint failure probability, P_{fij}.

Proposed Point Estimate of Joint Failure Probability

To express the formulas more conveniently, β_1 and β_2 are used to represent β_i and β_j, respectively, and ρ is used to represent ρ_{ij}. Without loss of generality, assume $0<\beta_1\leqslant\beta_2$.

Let Z_1 and Z_2 be the limit state functions in standard normal space corresponding to β_1 and β_2; then, the geometrical relationship between $Z_1=0$ and $Z_2=0$ can be depicted in Fig. 1(a) when $\beta_1/\beta_2\geqslant\rho$ and in Fig. 1(b) when $\beta_1/\beta_2<\rho$.

Let the angle between OB (β_1) and OC (β_2) be υ; then

$$\nu=\arccos(\rho) \tag{12}$$

In Fig. 1, the crossing point of $Z_1=0$ and $Z_2=0$ is point A. Define the length of the line segment OA as crossing index β_0, and denote the angle between OA and OB as υ_1 and the angle between OA and OC as υ_2; then υ_1 and υ_2 can be expressed as

$$\nu_1=\arccos(\beta_1/\beta_0) \tag{13a}$$

$$\nu_2=\arccos(\beta_2/\beta_0) \tag{13b}$$

With the aid of the geometrical relationships of β_0, β_1, and β_2, β_0 can be given as (see Appendix I)

$$\beta_0=\sqrt{\frac{\beta_1^2-2\rho\beta_1\beta_2+\beta_2^2}{1-\rho^2}} \tag{14}$$

Let V_1 denote the area of the failure zone between ray OA and $Z_1=0$, and V_2 denote the area of the failure zone between OA and $Z_2=0$, shown as the respective shaded zones in Fig. 1. The angle $\angle$OAB is equal to $\pi/2-\upsilon_1$ and the angle $\angle$OAC is equal to $\pi/2-\upsilon_2$. Since the joint failure probability P_{f12} is the area of the failure zone between $Z_1=0$ and $Z_2=0$, P_{f12} can be given as the following equation according to the geometrical relations in Fig. 1:

$$P_{f12}=\begin{cases} V_1+V_2 & \beta_1/\beta_2\geqslant\rho \\ P_{f2}+V_1-V_2 & \beta_1/\beta_2<\rho \end{cases} \tag{15}$$

In particular, when $\beta_1/\beta_2=\rho$, $\beta_0=\beta_2$. Then one can see that $V_2=P_{f2}/2$ from both Fig. 1(a) and Fig. 1(b), which means that both the formulas in Eq. (15) give the same results for $\beta_1/\beta_2=\rho$.

When $\upsilon_m\geqslant\pi/4$ (where $m=1,2$), V_m of Eq. (15) can be obtained by constructing two perpendicular lines, DD′ and EE′, through point A; the crossing point of $Z_1=0$ and $Z_2=0$. Let the angle $\angle$DAO=$\angle$EAO=$\pi/4$, as shown in Fig. 2.

Obviously, if EE′ and DD′ are considered to be limit state lines, both of their corresponding reliability indices would be $\beta_0/\sqrt{2}$. Since EE′ and DD′ are perpendicular, the probability associated with the area enclosed by $\angle$E′AD′ (gray area in Fig. 2) can be obtained as $\Phi^2(-\beta_0/\sqrt{2})$. Since the angle between OA and $Z_m=0$ (the shaded zone in Fig. 2) that corresponds to V_m is $\pi/2$-υ_m, we have

$$\frac{\pi/2}{\Phi^2(-\beta_0/\sqrt{2})}\approx\frac{\pi/2-\nu_m}{V_m} \tag{16}$$

Hence V_m can be given as

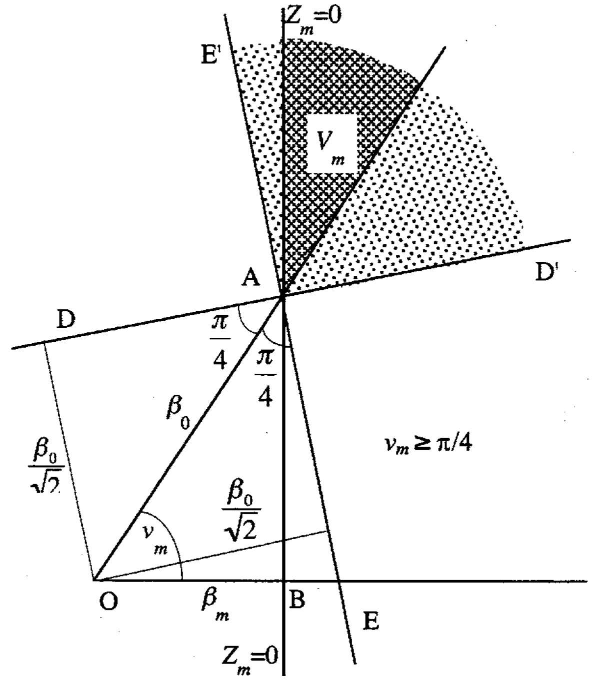


Fig. 2. Geometrical relations for $\upsilon_m\geqslant\pi/4$

$$V_m\approx\Phi^2(-\beta_0/\sqrt{2})\left(1-\frac{2\nu_m}{\pi}\right);\quad \nu_m\geqslant\pi/4\quad\text{where } m=1,2 \tag{17}$$

When $\upsilon_m<\pi/4$ (where $m=1,2$), with the horizontal line FF′ through point A, the reliability index corresponding to limit state line FF′ is equal to $\sqrt{\beta_0^2-\beta_m^2}$ and the probability corresponding to the angle between $Z_m=0$ and FF′, shown as the gray area in Fig. 3(a), can be given as $\Phi(-\beta_m)\Phi(-\sqrt{\beta_0^2-\beta_m^2})$, because the lines $Z_m=0$ and FF′ are perpendicular. Denote the probability corresponding to the angle OA and FF′ as V' [the shaded zone in Fig. 3(a)], then V_m can be given by the following equation according to the geometrical relations in Fig. 3(a)

$$V_m\approx\Phi(-\beta_m)\Phi(-\sqrt{\beta_0^2-\beta_m^2})-V'$$

In order to obtain V', draw two perpendicular lines DD′ and EE′ through point A. Let the angle $\angle$DAO=$\angle$EAO=$\pi/4$, as shown in Fig. 3(b). Obviously, both of the reliability indices corresponding to the limit state lines EE′ and DD′ would be $\beta_0/\sqrt{2}$, and the probability corresponding to the area defined by the angle $\angle$E′AD′ [gray area in Fig. 3(b)] can be obtained as $\Phi^2(-\beta_0/\sqrt{2})$ because EE′ and DD′ are perpendicular.

Since the angle between OA and FF′ [the shaded zone in Fig. 3(b)] that corresponds to V' is equal to υ_m, one obtains

$$\frac{\pi/2}{\Phi^2(-\beta_0/\sqrt{2})}\approx\frac{\nu_m}{V'}$$

and then

$$V'\approx\Phi^2(-\beta_0/\sqrt{2})\frac{2\nu_m}{\pi}$$

Therefore

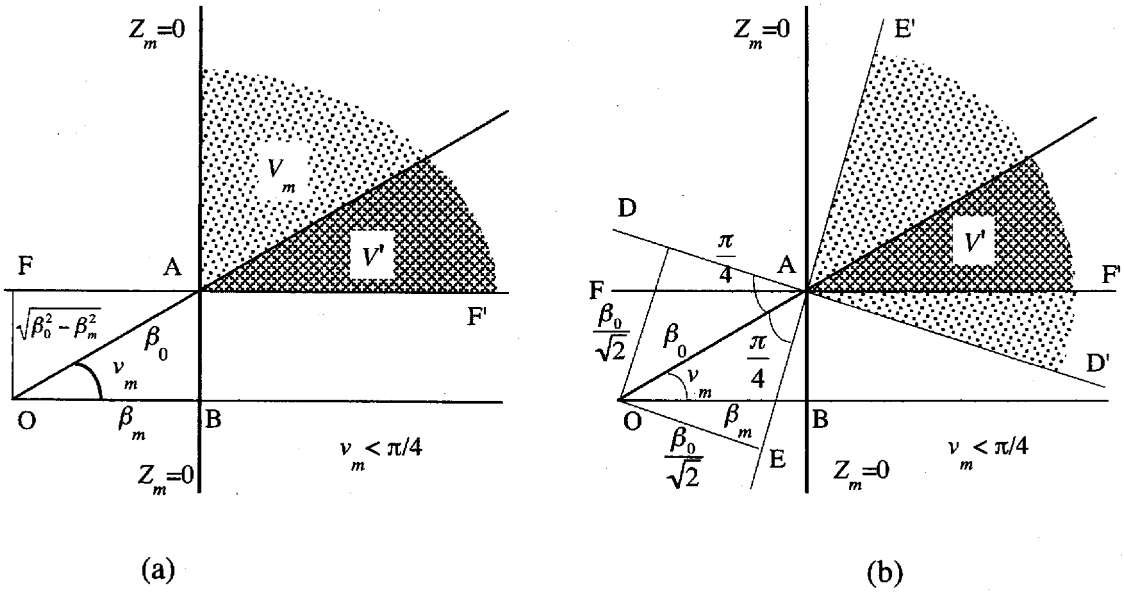


Fig. 3. Geometrical relations for $\upsilon_m<\pi/4$

$$V_m \approx \Phi(-\beta_m)\Phi(-\sqrt{\beta_0^2-\beta_m^2})-\Phi^2(-\beta_0/\sqrt{2})\frac{2\nu_m}{\pi};$$

$$\nu_m<\pi/4, \quad \text{where } m=1,2 \tag{18}$$

Write

$$P_m=\Phi(-\beta_m)\Phi(-\sqrt{\beta_0^2-\beta_m^2})$$
$$P_0=\Phi^2(-\beta_0/\sqrt{2}) \tag{19}$$

Then Eqs. ((17) and (18)) can be written as

$$V_m=\begin{cases} P_0\left(1-\dfrac{2\nu_m}{\pi}\right) & \nu_m \geqslant \dfrac{\pi}{4} \\ P_m-P_0\dfrac{2\nu_m}{\pi} & \nu_m<\dfrac{\pi}{4}\end{cases} \quad \text{where } m=1,2 \tag{20}$$

Eqs. (15) and (20) are the proposed formulas for estimating the joint failure probability P_{f12}.

When $\upsilon_m=\pi/4$, according to Eq. (13), $\beta_0=\sqrt{2}\beta_m$, then $P_m=P_0=\Phi^2(-\beta_m)$. In this case, the two formulas in Eq. (20) give the same results.

The formulas also can be written as follows: For $\beta_1/\beta_2\geqslant\rho$

$$P_{f12}=\begin{cases} P_2+P_0\left(1-\dfrac{2\nu}{\pi}\right) & \nu_1\geqslant\dfrac{\pi}{4} \\ P_1+P_2-P_0\dfrac{2\nu}{\pi} & \nu_1<\dfrac{\pi}{4}\end{cases} \tag{21}$$

And for $\beta_1/\beta_2<\rho$

$$P_{f12}=\begin{cases} P_{f2}-P_0\dfrac{2\nu}{\pi} & \nu_1\geqslant\dfrac{\pi}{4},\ \nu_2\geqslant\dfrac{\pi}{4} \\ P_{f2}-P_2+P_0\left(1-\dfrac{2\nu}{\pi}\right) & \nu_1\geqslant\dfrac{\pi}{4},\ \nu_2<\dfrac{\pi}{4} \\ P_{f2}+P_1-P_2-P_0\dfrac{2\nu}{\pi} & \nu_1<\dfrac{\pi}{4}\end{cases} \tag{22}$$

where

$$P_1=\Phi(-\beta_1)\Phi(-\sqrt{\beta_0^2-\beta_1^2}) \tag{23a}$$

$$P_2=\Phi(-\beta_2)\Phi(-\sqrt{\beta_0^2-\beta_2^2}) \tag{23b}$$

in which P_1 and P_2 can also be expressed as follows (see Appendix II):

$$P_1=\Phi(-\beta_1)\Phi\left(-\frac{\beta_2-\rho\beta_1}{\sqrt{1-\rho^2}}\right) \tag{24a}$$

$$P_2=\Phi(-\beta_2)\Phi\left(-\frac{|\beta_1-\rho\beta_2|}{\sqrt{1-\rho^2}}\right) \tag{24b}$$

Eqs. (24) are almost the same as P(A) and P(B) in Eq. (10) if one uses β_1, β_2, and ρ to represent β_i, β_j, and ρ_{ij}, respectively, in Ditlevsen's formula.

In particular, when $\rho=0$, it can be seen that $\upsilon=\arccos(0)=\pi/2$ and $\beta_0=\sqrt{\beta_1^2+\beta_2^2}$; then

$$P_1=P_2=\Phi(-\beta_1)\Phi(-\beta_2)$$

Since $\upsilon_1+\upsilon_2=\upsilon=\pi/2$, and $\upsilon_1\geqslant\upsilon_2$, one can see that $\upsilon_1\geqslant\pi/4$. Then P_{f12} is given by

$$P_{f12}=\Phi(-\beta_1)\Phi(-\beta_2); \quad \rho=0 \tag{25}$$

Another special case is when $\rho=1$. Obviously, P_{f12} cannot be directly given by the equations given above since β_0 is not defined when $\rho=1$. In this case, when $\rho\to1$, we have (see Appendix III)

$$\lim_{\rho\to1}P_{f12}=P_{f2} \tag{26}$$

Examination through Specific Examples

In order to evaluate the advantage or superiority of the proposed method, a number of series structural systems are examined.

Example 1

Consider first a series structural system with only two failure modes. Several cases are examined to compare the results by different methods as follows.

When the reliability indices of the two failure modes are the same, the results for P_{f12} are presented in Figs. 4(a–c) with $\beta_1 = \beta_2 = 2$; $\beta_1 = \beta_2 = 3$; and $\beta_1 = \beta_2 = 4$, respectively, and for various correlation coefficient ρ. Figs. 4(a–c) show the solutions obtained by integration, the Ditlevsen's bounds, Feng's point estimation, and the point estimation proposed here. From the figures, it can be seen that (1) Ditlevsen's bounds correctly bound the integration results; (2) the results by Feng's method are quite close to the integration results; and (3) the results given by the present method have a better agreement with the integration method than those by Feng's method.

When the reliability indices for the two failure modes are different, the variations of the joint failure probability P_{f12} with respect to the correlation coefficient ρ are depicted in Figs. 4(d–f), respectively, for $\beta_1 = 2.5$, $\beta_2 = 3.5$; $\beta_1 = 2$, $\beta_2 = 4$; and $\beta_1 = 1$, $\beta_2 = 5$. From these figures, it can be seen that the integration results are always located between the narrow bound solutions.

Both the results by Feng's method and the present method have good agreement with the integration method when $\rho = 0$ and $\rho = 1$. However, whereas the results by the present method have a good agreement with those by integration for all three cases, the results given by Feng's method tend to be lower than the lower bound solution, especially for large ρ, and this discrepancy becomes very large for significant differences in the two reliability indices [Fig. 4(f)].

The variations of the joint failure probability P_{f12} with respect to the difference between the reliability indices for two failure modes are depicted in Figs. 5(a and b), respectively, for $\beta_1 = 2$, $\rho = 0.5$, and for $\beta_1 = 2$, $\rho = 0.8$. From these figures, it can be seen that the larger the difference between the two reliability indices, the narrower the reliability bound width. Feng's method gives good results for the small difference between the two reliability indices, and the present method gives good approximation of the integration results for the whole investigation range.

Example 2

Consider next a series structural system with four failure modes, in which the first-order reliability indices for the four individual failure modes have been obtained as $\beta_1 = 2.5$, $\beta_2 = 2.5$, $\beta_3 = 3.0$, $\beta_4 = 3.5$, and the correlation coefficient between every pair of failure modes is assumed to be $\rho = 0.86$. The joint failure probability results, P_{fij}, calculated by different methods are listed in Table 1, and the corresponding results for the system failure probability, P_f, are listed in Table 2. From Tables 1 and 2, it can be seen that the results by the present method are between the lower and upper bounds and are in good agreement with those obtained by numerical integration. Also, the bounds obtained with the present method are the narrowest among all the methods.

Example 3

Consider a one-story one-bay elastoplastic frame shown in Fig. 6 [after Ono et al. (1990)]. The loads M_i and member strengths S_i are independent log-normal random variables with mean values of $\mu_{M1} = \mu_{M2} = 500$ ft kip, $\mu_{M3} = 667$ ft kip, $\mu_{S1} = 50$ kip, $\mu_{S2} = 100$ kip and standard deviations of $\sigma_{M1} = \sigma_{M2} = 75$ ft kip, $\sigma_{M3} = 100$ ft kip, $\sigma_{S1} = 15$ kip, $\sigma_{S2} = 10$ kip. The performance functions that correspond to the six most likely failure modes obtained from stochastic limit analysis are listed below, with the FORM reliability index for each mode given in parentheses to show the relative dominance of the different modes:

$$g_1 = 2M_1 + 2M_2 - 15S_1 \ (\beta_F = 3.247) \tag{27a}$$

$$g_2 = M_1 + 3M_2 + 2M_3 - 15S_1 - 10S_2 \ (\beta_F = 3.551) \tag{27b}$$

$$g_3 = 2M_1 + M_2 + M_3 - 15S_1 \ (\beta_F = 3.562) \tag{27c}$$

$$g_4 = M_1 + 2M_2 + M_3 - 15S_1 \ (\beta_F = 3.562) \tag{27d}$$

$$g_5 = M_1 + M_2 + 2M_3 - 15S_1 \ (\beta_F = 3.784) \tag{27e}$$

$$g_6 = M_1 + M_2 + 4M_3 - 15S_1 - 10S_2 \ (\beta_F = 3.848) \tag{27f}$$

Using the performance functions listed in Eq. (27), the correlation matrix is as follows:

$$[C] = \begin{bmatrix} 1 & 0.810 & 0.942 & 0.875 & 0.753 & 0.499 \\ 0.810 & 1 & 0.932 & 0.837 & 0.895 & 0.855 \\ 0.942 & 0.932 & 1 & 0.937 & 0.920 & 0.749 \\ 0.875 & 0.837 & 0.937 & 1 & 0.920 & 0.749 \\ 0.753 & 0.895 & 0.920 & 0.920 & 1 & 0.923 \\ 0.499 & 0.855 & 0.749 & 0.749 & 0.923 & 1 \end{bmatrix}$$

and the joint failure probability for each pair of failure modes are given in the following matrix:

$$[P_{fij}] = 10^{-6} \begin{bmatrix} 582.8 & 72.81 & 147.1 & 101.4 & 28.14 & 4.778 \\ 72.81 & 191.8 & 88.45 & 47.40 & 40.39 & 26.43 \\ 147.1 & 88.45 & 183.9 & 89.31 & 46.88 & 13.27 \\ 101.4 & 47.40 & 89.31 & 183.9 & 46.88 & 13.27 \\ 28.14 & 40.39 & 46.88 & 46.88 & 77.14 & 27.83 \\ 4.778 & 26.43 & 13.27 & 13.27 & 27.83 & 59.50 \end{bmatrix}$$

from which the lower and upper bounds of the system failure probability are obtained, respectively, as $7.017 \cdot 10^{-4}$ and $9.331 \cdot 10^{-4}$. The corresponding MCS solution using a 10-million sample size is $6.147 \cdot 10^{-4}$ with a COV of 1.275%. One can see that the MCS result is outside the indicated bounds. This is because the FORM reliability indices used in calculating the above bounds are not accurate for each performance function. Using the 4M approach (Zhao and Ang 2003), the reliability indices are more accurately obtained as 3.293, 3.623, 3.629, 3.629, 3.871, 3.957, corresponding to the six respective performance functions of Eq. (27). With these latter reliability indices, the joint failure probability for each pair of failure modes is then obtained as follows:

$$[P_{fij}] = 10^{-6} \begin{bmatrix} 495.1 & 56.0 & 115.4 & 79.51 & 20.28 & 3.080 \\ 56.0 & 145.2 & 66.66 & 35.14 & 28.49 & 17.28 \\ 115.4 & 66.66 & 142.3 & 68.13 & 33.46 & 8.628 \\ 79.51 & 35.15 & 68.13 & 142.3 & 33.46 & 8.628 \\ 20.28 & 28.49 & 33.46 & 33.46 & 54.23 & 18.06 \\ 3.080 & 17.28 & 8.628 & 8.628 & 18.06 & 37.86 \end{bmatrix}$$

from which the bounds of the system failure probability become $5.844 \cdot 10^{-4}$ and $7.147 \cdot 10^{-4}$. Then we can observe that the MCS solution for the system failure probability of $6.147 \cdot 10^{-4}$ is clearly bounded by the narrow bounds.

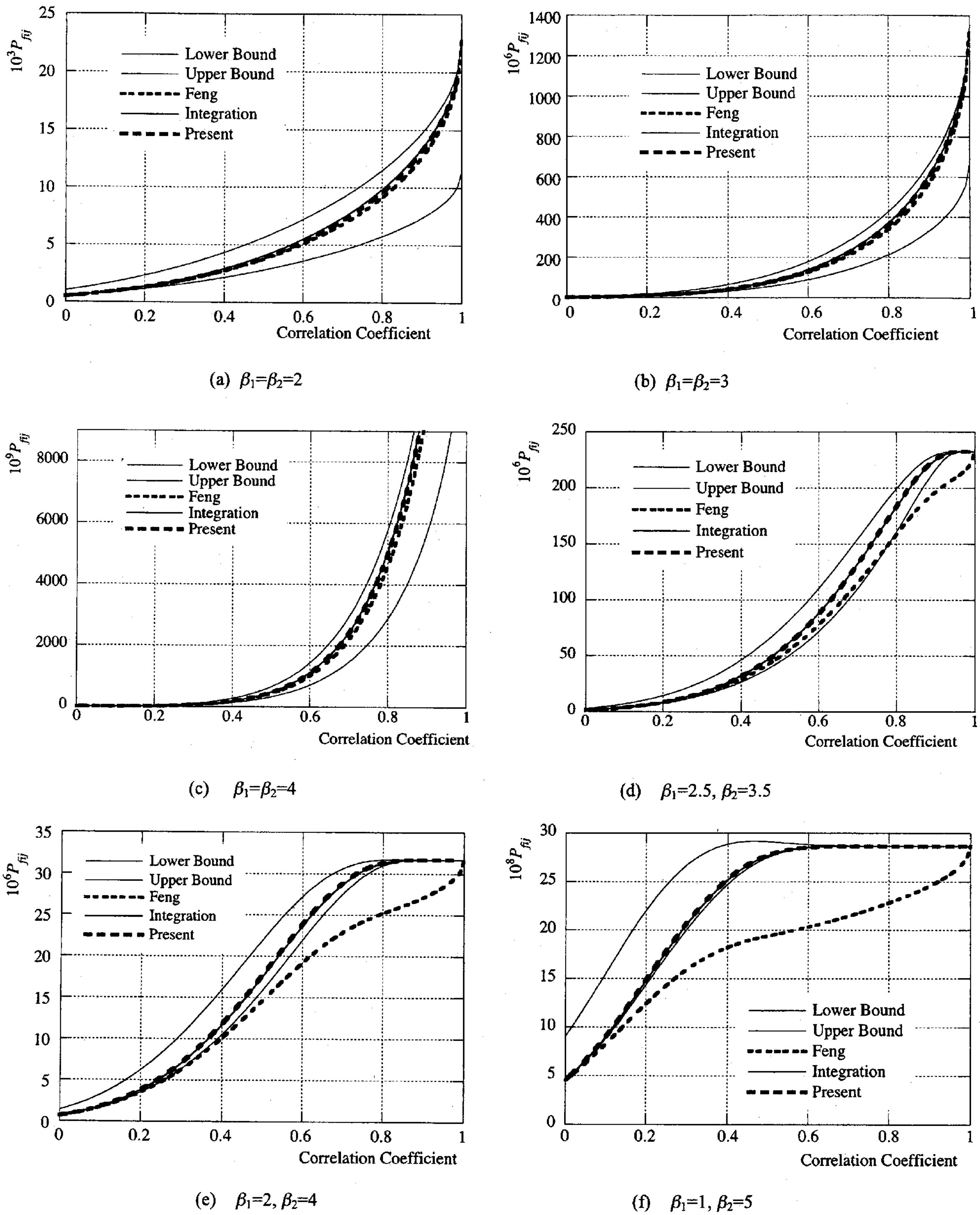
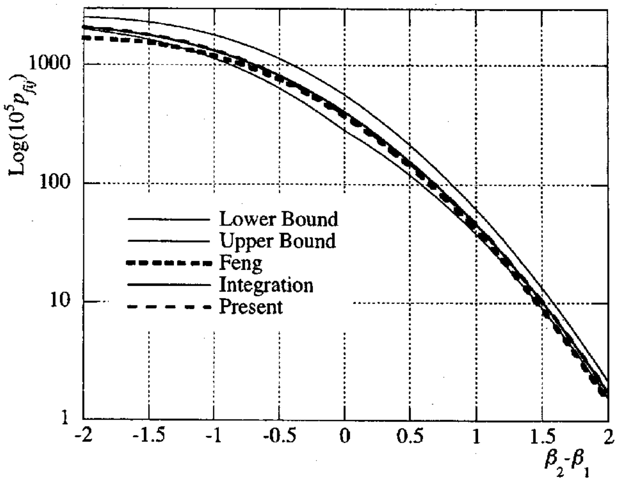
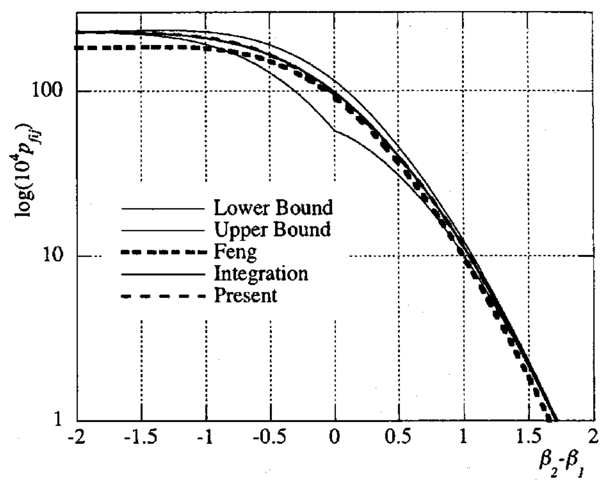


Fig. 4. Variations of joint failure probability P_{f12} with respect to correlation coefficient



(a) ρ=0.5, β_1=2



(b) ρ=0.8, β_1=2

Fig. 5. Variations of joint failure probability P_{f12} with difference between two reliability indices

Example 4

Finally, consider the simple elastoplastic beam-cable system shown in Fig. 7 [after Ang and Tang 1984]. The performance functions of the potential failure modes are listed below with the respective FORM reliability indices indicated in parentheses

$$g_1 = 6M - L^2/2 \; (\beta_F = 3.322) \tag{28a}$$

$$g_2 = F_1 L + 2F_2 L - 2wL^2 \; (\beta_F = 3.647) \tag{28b}$$

$$g_3 = M + F_2 L - wL^2/2 \; (\beta_F = 4.515) \tag{28c}$$

$$g_4 = 2M + F_1 L - wL^2 \; (\beta_F = 4.515) \tag{28d}$$

where M, F_1, F_2, and w are normally distributed with mean values of μ_w=2 kip/ft, μ_{F1}=60 kip, μ_{F2}=30 kip, and μ_M =100 ft kip and COVs of V_w=0.2 and V_F=V_M=0.1.

Using the performance functions listed in Eq. (28), the correlation matrix is obtained as

$$[C] = \begin{bmatrix} 1 & 0.412 & 0.534 & 0.534 \\ 0.412 & 1 & 0.856 & 0.856 \\ 0.534 & 0.856 & 1 & 0.553 \\ 0.534 & 0.856 & 0.553 & 1 \end{bmatrix}$$

and the joint failure probability of each pair of failure modes are given in the following matrix

$$[P_{fij}] = 10^{-6} \begin{bmatrix} 446.6 & 3.853 & 0.542 & 0.542 \\ 3.853 & 132.6 & 1.324 & 1.324 \\ 0.542 & 1.324 & 3.163 & 0.035 \\ 0.542 & 1.324 & 0.035 & 3.163 \end{bmatrix}$$

from which the bounds of the system failure probability are $5.779 \cdot 10^{-4}$ and $5.790 \cdot 10^{-4}$. The result obtained by numerical integration is $5.780 \cdot 10^{-4}$. One can see that the bound width is quite narrow, and both the lower and upper bounds are close to the solution obtained through numerical integration.

Conclusion

In order to improve the narrow bounds of the failure probability of a series structural system, a point estimation method is proposed for calculating the joint probability of every pair of failure modes of the system. Based on the computational results of the illustrative examples, the following conclusions can be observed:

1. With the proposed point estimation of the failure probability for each pair of failure modes in a series structural system, the results of the narrow bound method can be improved.
2. When the correlation coefficient, ρ=0, or ρ=1, the proposed method gives accurate solutions, whereas when $0<\rho<1$, the present method yields results that are quite close to those obtained by numerical integration and are consistently located between the lower and upper bound solutions.
3. Sometimes accurate estimates of the reliability indices of the

Table 1. Calculation of Joint Failure Probability $P_{fij}(10^{-4})$

Method	Present method	Feng's method	Ditlevsen's method		Numerical integration
			Lower bound	Upper bound	
P_{f21}	25.961	25.385	15.301	30.601	27.340
P_{f31}, P_{f32}	9.5369	8.7634	7.5905	10.5640	9.6690
P_{f41}, P_{f42}	2.1401	1.8334	1.9596	2.2101	2.1150
P_{f43}	1.5130	1.3802	1.1813	1.6638	1.5270

Table 2. Calculation Results of System Failure Probability $P_f(10^{-3})$

Method	Present method	Feng's method	Ditlevsen's method	Numerical integration
Lower bound	9.8234	9.8809	9.3593	
Upper bound	10.2386	10.4040	11.5170	P_f=9.8912
Bound width	0.4152	0.5231	2.1577	

individual failure modes is necessary for determining the correct narrow bounds of the system failure probability. This is illustrated in Example 3.

4. The method of Feng(1989) gives good results when the reliability indices for the pair of failure modes are the same; however, when the reliability indices of the two failure modes are different, Feng's method generally gives results that are below the lower bound for a relatively large correlation coefficient. Moreover, this error increases for a larger difference in the two reliability indices.

Appendix I. Crossing Index β_0

According to Fig. 1, $|OA|=\beta_0$, $|OB|=\beta_1$, $|OC|=\beta_2$.

In the triangle ΔOBC, according to the cosine law

$$|BC|^2 = |OB|^2 - 2(\cos\angle BOC)|OB||OC| + |OC|^2$$
$$= \beta_1^2 - 2\cos\nu\cdot\beta_1\beta_2 + \beta_2^2 = \beta_1^2 - 2\rho\beta_1\beta_2 + \beta_2^2$$

$$\because \angle OCA = \angle OBA = \pi/2$$

$\therefore$ Points O, A, B, C lie on the same circle with the center of the circle being at the midpoint of line segment OA.

$\because$ In the same circle, $\angle OAB = \angle OCB$.

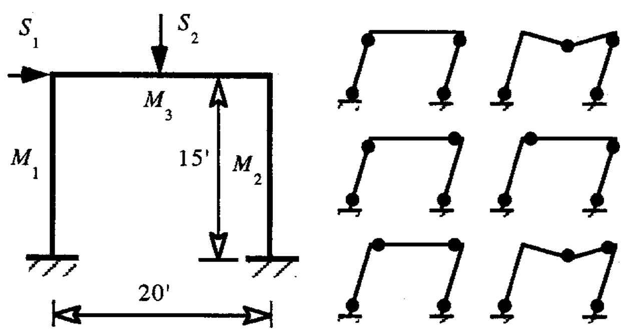


Fig. 6. One-story, one-bay frame of Example 3

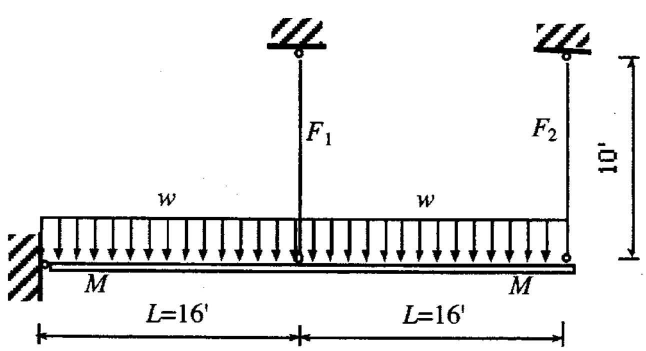


Fig. 7. Beam-cable system of Example 4

In the triangle ΔOAB, according to the sine law

$$\frac{\beta_0}{\sin\pi/2} = \frac{\beta_1}{\sin\angle OAB}$$

In the triangle ΔOCB, according to the sine law

$$\frac{\beta_1}{\sin\angle OCB} = \frac{|BC|}{\sin\nu}$$

$$\because \angle OAB = \angle OCB$$

$$\therefore \frac{\beta_0}{\sin\pi/2} = \frac{|BC|}{\sin\nu}$$

$$\therefore |BC|^2 = \beta_0^2\sin^2\nu = \beta_0^2(1-\cos^2\nu) = \beta_0^2(1-\rho^2)$$

$$\therefore \beta_1^2 - 2\rho\beta_1\beta_2 + \beta_2^2 = \beta_0^2(1-\rho^2)$$

$$\therefore \beta_0 = \sqrt{\frac{\beta_1^2 - 2\rho\beta_1\beta_2 + \beta_2^2}{1-\rho^2}}$$

Appendix II

In the present paper, according to Eq. (14)

$$\beta_0^2 - \beta_2^2 = \frac{\beta_1^2 - 2\rho\beta_1\beta_2 + \beta_2^2}{1-\rho^2} - \beta_2^2 = \frac{1}{1-\rho^2}(\beta_1 - \rho\beta_2)^2$$

Then

$$\sqrt{\beta_0^2 - \beta_2^2} = \frac{|\beta_1 - \rho\beta_2|}{\sqrt{1-\rho^2}}$$

Hence

$$P_2 = \Phi(-\beta_2)\Phi\left(-\frac{|\beta_1 - \rho\beta_2|}{\sqrt{1-\rho^2}}\right)$$

Similarly

$$P_1 = \Phi(-\beta_1)\Phi\left(-\frac{\beta_2 - \rho\beta_1}{\sqrt{1-\rho^2}}\right)$$

Appendix III. Limit of P_{f12} for $\rho\to1$

When $\rho=1$, according to Eq. (12), one has $\lim_{\rho\to1}\nu = \lim_{\rho\to1}\arccos(\rho)=0$

1. If $\beta_1/\beta_2=1$, one can see that $\nu_1+\nu_2=\nu=0$, and $\beta_0=\beta_1=\beta_2$, then according to Eq. (19), one has

$$P_1 = \Phi(-\beta_1)\Phi(0) = \frac{1}{2}\Phi(-\beta_1)$$

$$P_2 = \Phi(-\beta_2)\Phi(0) = \frac{1}{2}\Phi(-\beta_2) = P_1$$

Therefore

$$P_{f12} = P_1 + P_2 - P_0\cdot\frac{2\nu}{\pi} = \Phi(-\beta_2) = P_{f2}$$

2. If $\beta_1/\beta_2 < 1$, according to Eq. (14), one can see that when $\rho \to 1$, $\beta_0 \to \infty$, and hence $\sqrt{\beta_0^2 - \beta_1^2} \to \infty$, $\sqrt{\beta_0^2 - \beta_2^2} \to \infty$, and $\beta_0/\sqrt{2} \to \infty$.

According to Eq. (19)

$$\lim_{\rho\to 1} P_1 = \lim_{\rho\to 1} \Phi(-\beta_1)\Phi(-\sqrt{\beta_0^2 - \beta_1^2}) = 0;$$

$$\lim_{\rho\to 1} P_2 = \lim_{\rho\to 1} \Phi(-\beta_2)\Phi(-\sqrt{\beta_0^2 - \beta_1^2}) = 0; \text{ and}$$

$$\lim_{\rho\to 1} P_0 = \lim_{\rho\to 1} \Phi^2(-\beta_0/\sqrt{2}) = 0.$$

Therefore, according to Eq. (22), one obtains $\lim_{\rho\to 1} P_{f12} = P_{f2}$.

References

Adduri, R. P., Penmestsa, R., and Grandhi, R. (2004). "Estimation of structural system reliability using fast Fourier transforms." *10th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conf.*, Albany, New York.

Ang, A. H-S., and Tang, W. (1984). *Probability concepts in engineering planning and design. II: Decision, risk, and reliability*, Wiley, New York.

Cornell, C. A. (1967). "Bounds on the reliability of structural systems." *J. Struct. Div.*, 93(1), 171–200.

Ditlevsen, O. (1979). "Narrow reliability bounds for structural systems." *J. Struct. Mech.*, 7(4), 453–472.

Feng, Y. S. (1989). "A method for computing structural system reliability with high accuracy." *Comput. Struct.*, 33(1), 1–5.

Ono, T., Idota, H., and Dozuka, A. (1990). "Reliability evaluation of structural systems using higher-order moment standardization technique." *J. Struct. Constr. Eng.*, 418, 71–79 (in Japanese).

Penmestsa, R., and Grandhi, R. (2002). "Structural system reliability quantification using multipoint function." *AIAA J.*, 40(12), 2526–2531.

Song, B. F. (1992). "A numerical integration method in affine space and a method with high accuracy for computing structural system reliability." *Comput. Struct.*, 42(2), 255–262.

Wu, Y. T., and Burnside, O. H. (1990). "Computational methods for probability of instability calculations." *AIAA/ASME/ASCE/AS/ASC 31st Structures, Structural Dynamics and Materials Conf.*, A90–29283 11–39.

Zhao, Y. G., and Ang, A. H-S. (2003). "System reliability assessment by method of moments." *J. Struct. Eng.*, 129(10), 1341–1349.