Third-Moment Standardization for Structural Reliability Analysis

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Abstract: First- and second-order reliability methods are generally considered to be among the most useful for computing structural reliability. In these methods, the uncertainties included in resistances and loads are generally expressed as continuous random variables that have a known cumulative distribution function. The Rosenblatt transformation is a fundamental requirement for structural reliability analysis. However, in practical applications, the cumulative distribution functions of some random variables are unknown, and the probabilistic characteristics of these variables may be expressed using only statistical moments. In the present study, a structural reliability analysis method with inclusion of random variables with unknown cumulative distribution functions is suggested. Normal transformation methods that make use of high-order moments are investigated, and an accurate third-moment standardization function is proposed. Using the proposed method, the normal transformation for random variables with unknown cumulative distribution functions can be realized without using the Rosenblatt transformation. Through the numerical examples presented, the proposed method is found to be sufficiently accurate to include the random variables with unknown cumulative distribution functions in the first- and second-order reliability analyses with little extra computational effort.

INTRODUCTION

The structural reliability problem is often formulated in terms of a vector of basic random variables $\mathbf{X} = [x_1, \ldots, x_n]^T$, which represent uncertain quantities, such as loads, environmental factors, material properties, structural dimensions and variables introduced in order to account for modeling and prediction errors, and the performance function $G(\mathbf{X})$ describing the limit state of the structure in terms of $\mathbf{X}$. The probability of failure is given by the n-fold integral

$$P_f = \int_{G(\mathbf{X})\le 0} f(\mathbf{X})\, d\mathbf{X} \tag{1}$$

in which $f(\mathbf{X})$ = joint probability density function (PDF) of $\mathbf{X}$.

Difficulty in computing this probability has led to the development of various approximations (Madsen et al. 1986). The first-order reliability method (FORM), a full-distribution reliability method, is considered to be among the most useful for computing structural reliability (Bjerager 1991). Over the past three decades, contributions from numerous studies have brought FORM to fruition as a reliability method (Hasofer and Lind 1974; Rackwitz 1976; Shinozuka 1983), and many reliability methods based on FORM have been developed. These include the second-order reliability method (SORM) (Breitung 1984; Der Kiureghian et al. 1987; Der Kiureghian and De Stefano 1991; Cai and Elishakoff 1994), importance sampling Monte Carlo simulation (Melchers 1990; Fu 1994), first-order third-moment reliability method (FOTM) (Tichy 1994), and response surface approach (Rajashekhar and Ellingwood 1993; Liu and Moses 1994). In almost all of these methods, the basic random variables are assumed to have a known cumulative distribution function (CDF) or probability density function (PDF). Based on PDF/CDF, the normal transformation (x-u transformation) and its inverse transformation (u-x transformation) are realized by using the Rosenblatt transformation (Hohenbichler and Rackwitz 1981). In reality, however, due to the lack of statistical data, the PDF/CDFs of some basic random variables are often unknown, and the probabilistic characteristics of these variables are often expressed using only statistical moments. Under such a condition, the Rosenblatt transformation cannot be applied, and a strict evaluation of the probability of failure is not possible. Thus, an alternative measure of reliability is required.

The first thorough study on structural reliability under incomplete probability information was performed by Der Kiureghian and Liu (1986). Most previous studies of this subject dealt with second-moment methods, in which only the mean values and standard deviations of the basic random variables are known (Ang and Cornell 1974; Hasofer and Lind 1974; Ditlevsen 1979; Madsen et al. 1986). In such a case, the variables are commonly transformed into a set of standard variables having zero means and unit standard deviation. A comprehensive framework for the analysis of structural reliability under incomplete probability information was proposed by Der Kiureghian and Liu (1986), in which incomplete probability information on random variables, including moments, bounds, marginal distributions, and partial joint distributions, is incorporated in reliability analysis under stipulated requirements of consistency, invariance, operability, and simplicity. The proposed method was found to be consistent with full distribution structural reliability theories and has been used to measure structural safety under imperfect states of knowledge (Der Kiureghian 1989).

When the PDF/CDF of a random variable is unknown, an approach based on the Bayesian idea, in which the distribution is assumed to be a weighted average of all candidate distributions. In this type of modeling, the weights represent the subjective probabilities of each candidate distribution being the true distribution, as suggested by Der Kiureghian and Liu (1986). For a variable x_1 with k candidate distributions $F_{1i}(x_1)$, $i = 1, \ldots, k$, the Bayesian distribution is written in the form of

$$F_1(x_1) = \sum_{i=1}^{k} p_{1i}F_{1i}(x_1) \tag{2}$$

in which p_{1i}, satisfying $\sum_{i=1}^{k} p_{1i} = 1$, are the weights. Furthermore, all candidate distributions are assumed to have the same mean and variance because these are assumed to be known quantities.

After obtaining the distribution in (2), the reliability analysis can be conducted in a manner similar to the full distribution structural reliability theories such as FORM/SORM. However,

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the problem arises as to how to select the candidate distributions and the weights. Recently, a method of estimating complex distributions using B-spline functions has been proposed (Zong and Lam 1998), in which the estimation of PDF is summarized as a nonlinear programming problem.

In FORM/SORM, the PDF/CDF is only used to determine the *x-u* and *u-x* transformations. Therefore, if the transformation functions can be obtained by other means, FORM/SORM will be possible without using PDF/CDF. In the present paper, for random variables with unknown PDF/CDF, the *x-u* and *u-x* transformation functions will be directly built using statistical moments, which are generally available from the statistical data of the random variables.

x-u AND *u-x* TRANSFORMATIONS USING HIGH-ORDER MOMENTS

Edgeworth and Cornish-Fisher Expansions

For a random variable x with CDF F and a standard normal random variable u, let

$$F(x) = \Phi(u) \tag{3}$$

where Φ is the CDF of u and the Rosenblatt transformation is used to solve this equation for u in terms of x, or for x in terms of u.

In order to realize the *x-u* and *u-x* transformations without using the PDF/CDF of x, the Edgeworth and Cornish-Fisher expansions (Stuart and Ord 1987) can be applied for cases of mild nonnormality (Appendix I). For a standardized random variable

$$x_s = \frac{x - \mu}{\sigma} \tag{4}$$

in which μ and σ are the mean value and standard deviation, respectively, of x, the *x-u* transformation is approximately given by

$$u = \Phi^{-1}\left\{\Phi(x_s) - \phi(x_s)\left[\frac{1}{6}\alpha_3 h_2 + \frac{1}{24}(\alpha_4 - 3)h_3 + \frac{1}{72}\alpha_3^2 h_5 + \frac{1}{1{,}296}h_8\alpha_3^3 + \frac{1}{1{,}152}h_7(\alpha_4 - 3)^2 + \frac{1}{1{,}728}h_9\alpha_3^2(\alpha_4 - 3) + \frac{1}{31{,}104}h_{11}\alpha_3^4\right]\right\} \tag{5}$$

using the first four polynomials of the Edgeworth expansion (Appendix I)

$$\begin{aligned} u = {} & x_s - \frac{1}{6}\alpha_3(x_s^2 - 1) - \frac{1}{24}(\alpha_4 - 3)(x_s^3 - 3x_s) \\ & + \frac{1}{36}\alpha_3^2(4x_s^3 - 7x_s) + \frac{1}{144}\alpha_3(\alpha_4 - 3)(11x_s^4 - 42x_s^2 + 15) \\ & - \frac{1}{648}\alpha_3^3(69x_s^4 - 187x_s^2 + 52) \\ & + \frac{1}{384}(\alpha_4 - 3)^2(5x_s^5 - 32x_s^3 + 35x_s) \\ & - \frac{1}{864}\alpha_3^2(\alpha_4 - 3)(111x_s^5 - 547x_s^3 + 456x_s) \\ & + \frac{1}{7{,}776}\alpha_3^4(948x_s^5 - 3{,}628x_s^3 + 2{,}473x_s) \end{aligned} \tag{6}$$

and using the first four polynomials of the inverse Cornish-Fisher expansion (Appendix I). In (5) and (6), h_j, $j = 1, 2, \ldots, 11$ is the jth Hermite polynomial, and α_3 and α_4 are the 3rd and 4th dimensionless central moments (the 1st and 2nd moment ratios), respectively. These parameters are in fact the skewness and kurtosis, respectively, of x_s, which, according to the definition of probability moments, are equal to those of x.

The *u-x* transformation is approximately given by

$$\begin{aligned} x_s = {} & u + \frac{1}{6}\alpha_3(u^2 - 1) + \frac{1}{24}(\alpha_4 - 3)(u^3 - 3u) \\ & - \frac{1}{36}\alpha_3^2(2u^3 - 5u) - \frac{1}{24}\alpha_3(\alpha_4 - 3)(u^4 - 5u^2 + 2) \\ & + \frac{1}{324}\alpha_3^3(12u^4 - 53u^2 + 17) \\ & - \frac{1}{384}(\alpha_4 - 3)^2(3u^5 - 24u^3 + 29u) \\ & + \frac{1}{288}\alpha_3^2(\alpha_4 - 3)(14u^5 - 103u^3 + 107u) \\ & - \frac{1}{7{,}776}\alpha_3^4(252u^5 - 1{,}688u^3 + 1{,}511u) \end{aligned} \tag{7}$$

using the first four polynomials of the Cornish-Fisher expansion (Appendix I).

An alternative *u-x* transformation can be derived by Hermite models (Winterstein 1985; Winterstein and Bjerager 1987), where x_s is expanded into Hermite series. A transformation somewhat analogous to the Cornish-Fisher expansion (Winterstein 1985) and a formula that has the same form as the first three terms of (7) have been proposed by Winterstein (1985).

Eqs. (5)–(7) give the explicit *u-x* and *x-u* transformations without using the PDF/CDF. The investigations presented in this paper will show that, although (5)–(7) are quite complicated, their accuracy is not better than the method developed in the present paper.

High-Order Moment Standardization Technique

Another rational and practical approach in this case is to build direct *x-u* and *u-x* transformation functions using the high-order moment standardization technique (HOMST) (Ono and Idota 1986), which involves the use of the following transformation:

$$u = S_x(x) = \sum_{j=1}^{k} a_j x^{j-1} \tag{8}$$

where a_j, $j = 1, \ldots, k$, are deterministic coefficients that are obtained by setting the first k central moment of $S_x(x)$ equal to those of the standard normal random variable.

In a particularly common case, the third-moment standardization function for the standardized random variable x_s in (4) is assumed to be (Ono and Idota 1986)

$$y = x_s + cx_s^2 \tag{9}$$

$$u = \frac{y - \mu_y}{\sigma_y} \tag{10}$$

in which c is determined by setting the skewness of y equal to that of the normal random variable, that is, c can be determined using the following equation:

$$\alpha_{3y}\sigma_y^3 = (\alpha_{6x} - 3\alpha_{4x} + 2)c^3 + 3(\alpha_{5x} - 2\alpha_{3x})c^2 + 3(\alpha_{4x} - 1)c + \alpha_{3x} = 0 \tag{11}$$

Since the use of the 5th and 6th moments is uncommon in engineering, assuming $|c| << 1$ according to the investigation by Ono and Idota (1986) in the cases of mild nonnormality, an approximate third-moment standardization function is given as (Zhao and Ono 1999b)

$$u = \frac{1}{a}(\alpha_3 + 3(\alpha_4 - 1)x_s - \alpha_3 x_s^2) \tag{12}$$

where

$$a = \sqrt{(9\alpha_4 - 5\alpha_3^2 - 9)(\alpha_4 - 1)} \tag{13}$$

Using (12), the approximate *u*-*x* transformation function based on third-moment standardization is obtained as

$$x_s = \frac{1}{2\alpha_3}\left[3(\alpha_4 - 1) - \sqrt{9(\alpha_4 - 1)^2 + 4\alpha_3(\alpha_3 - ua)}\right] \tag{14}$$

When the assumption $|c| << 1$ in (12) and (14) is not satisfied, the equations will not be accurate.

Accurate Third-Moment Standardization Function

Using (8) to obtain the kth moment standardization function, the $k(k - 1)$th central moment of x must be determined. Even for the third-moment standardization, the first six moments of x must be determined, and, as such, the standardization becomes complicated and obtaining the accurate standardization function becomes difficult. Since the *x*-*u* and *u*-*x* transformations form a pair, one transformation can be obtained from the other. In the present paper, the transformations are built from a *u*-*x* transformation that is assumed to be in the following form:

$$x_s = S_u(u) = \sum_{j=1}^{k} a_j u^{j-1} \tag{15}$$

where a_j, $j = 1, \ldots, k$, are deterministic coefficients that are obtained by making the first k central moment of $S_u(u)$ to be equal to that of x_s.

Using (15), to obtain the kth moment standardization function, only the first k central moments of x_s are needed. For the third-moment standardization, the *u*-*x* transformation is expressed as

$$x_s = S_u(u) = a_1 + a_2 u + a_3 u^2 \tag{16}$$

Making the first three central moments of $S_u(u)$ equal to those of x_s, a_1, a_2, and a_3 are obtained as (Appendix II)

$$a_3 = -a_1 = \pm\sqrt{2}\cos\left[\frac{\pi + |\theta|}{3}\right] \tag{17}$$

$$a_2 = \sqrt{1 - 2a_3^2} \tag{18}$$

where

$$\theta = \arctan\left(\frac{\sqrt{8 - \alpha_3^2}}{\alpha_3}\right) \tag{19}$$

The signs in (17) are taken to be the sign of α_3 (Appendix II).

From (19), in order to make (16) operable, α_3 should be limited in the range of

$$-2\sqrt{2} \le \alpha_3 \le 2\sqrt{2} \tag{20}$$

Skewnesses of some commonly used random variables are listed in Table 1. These values show that almost all the skewnesses are in the range of (20); that is to say, (16) is generally operable in engineering.

Particularly if $\alpha_3 = 0$, then $|\theta|$ is obtained as $\pi/2$, and a_1, a_2, and a_3 will be obtained as $a_1 = a_3 = 0$ and $a_2 = 1$ and the *u*-*x* transformation function will then degenerate as $x_s = u$.

From (16), the *x*-*u* transformation is readily obtained as

TABLE 1. Skewness of Some Common Distributed Random Variables

Distribution (1)	Coefficient of variation (2)	Skewness (3)
Lognormal	0.1	0.301
	0.2	0.608
	0.4	1.264
	0.6	2.016
	0.7	2.443
Exponential	1.0	2
Gumbel	—	1.140
Gamma	0.1	0.2
	0.2	0.4
	0.4	0.8
	0.6	1.2
	0.7	1.4
Frechet	0.1	1.662
	0.2	2.353
	0.3	3.353
	0.4	5.006
Weibull	0.1	−0.715
	0.2	−0.352
	0.3	−0.026
	0.4	0.277
	0.5	0.566
	0.7	1.131

$$u = \frac{-a_2 + \sqrt{a_2^2 - 4a_3(a_1 - x_s)}}{2a_3} \tag{21}$$

Unlike (12) and (14), obtaining the coefficients and (16) and (21) does not require the application of any approximation or assumption. Therefore, (21) is an accurate third-moment standardization function.

RELIABILITY ANALYSIS INCLUDING VARIABLES WITH UNKNOWN CDF

Using the first three moments of an arbitrary random variable x (continuous or discontinuous) with unknown PDF/CDF, a standard normal random variable u can be obtained using (21), and a random variable x' corresponding to u can be obtained from (16). Since u is a continuous random variable, x' will also be a continuous random variable. Although x and x' are different random variables, they correspond to the same standard normal random variable and have the same third central moment and the same statistical information source. Therefore, $f(x')$ can be considered to be an anticipated PDF of x. Using this PDF, (1) will become operable. Because the *u*-*x* and *x*-*u* transformations are realized directly by using (16) and (21), the specific form of $f(x')$ is not required in FORM/SORM. Assuming that the random variables with an unknown PDF/CDF are independent of those that have a PDF/CDF and are independent of each other as well, from (16) and (21), the element of the Jacobian matrix corresponding to a random variable x with an unknown PDF/CDF can be given by

$$J_{ii} = \frac{\partial u_i}{\partial x_i} = \frac{1}{\sigma(a_2 + 2a_3 u_i)} \tag{22}$$

For a reliability analysis with all the random variables that have a known PDF/CDF, the analysis can be conducted using the general FORM procedure (Ang and Tang 1984). For reliability analysis including random variables with an unknown PDF/CDF, the procedure can be rewritten as follows:

1. Divide the random variables $\mathbf{X}$ into two groups $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2]$, where $\mathbf{X}_1$ are the random variables that have a known PDF/CDF, and $\mathbf{X}_2$ those with unknown PDF/CDF.

2. Select an initial design point $\mathbf{X}^0$ (generally the mean value point).
3. Transform $\mathbf{X}^0$ into a standard normal space to obtain $\mathbf{U}^0$. For $\mathbf{X}_1$, the normal transformation is conducted using the Rosenblatt transformation, and for $\mathbf{X}_2$, the normal transformation is conducted using the third-moment standardization function (21).
4. Compute the derivatives of the performance function $G(\mathbf{X})$ to $\mathbf{X}$ at $\mathbf{X}^0$.
5. Transform the derivatives to $\mathbf{X}$ into those to $\mathbf{U}$ at $\mathbf{U}^0$ using the Jacobian matrix. For $\mathbf{X}_1$, the elements of the Jacobian matrix are obtained from the Rosenblatt transformation, and for $\mathbf{X}_2$, they are obtained using (22).
6. Obtain a new design point and compute the first-order reliability index.
7. Substituting $\mathbf{X}^{k+1}$ for $\mathbf{X}^0$ in step 3, repeat steps 3 through 6 until convergence.

The procedure is identical to that of the general FORM, with the exception of the conduction of the normal transformation and the computation of the elements of the Jacobian matrix corresponding to the random variables with unknown PDF/CDFs. Therefore, the reliability analysis with the inclusion of random variables with unknown PDF/CDFs using the proposed method requires few extra computational efforts, compared to the general FORM procedure.

When random variables that have an unknown PDF/CDF are contained in a performance function with strong nonlinearity, for which a more accurate method such as SORM is required, the proposed method can also be applied. In such a case, the computational procedure is identical to that of general SORM with the exception of the *u*-*x* and *x*-*u* transformations and the computation of the elements of the Jacobian matrix corresponding to the random variables with unknown PDF/CDFs.

EXAMPLES AND INVESTIGATIONS

u-*x* and *x*-*u* Transformations for Gamma Random Variable

In order to investigate the efficiency of the proposed third-moment standardization function, some random variables, for which the exact *u*-*x* and *x*-*u* transformations can be obtained, are selected. Example 1 considers a gamma random variable that has the following PDF:

$$f(x) = \frac{1}{\Gamma(\lambda)} e^{-x} x^{\lambda-1}, \quad 0 \le x < \infty, \quad \lambda > 0 \tag{23}$$

For $\lambda = 3$, the mean value, standard deviation, skewness, and kurtosis are obtained as $\mu = 3$, $\sigma = \sqrt{3}$, $\alpha_3 = 2\sqrt{3}/3 \approx 1.1547$, and $\alpha_4 = 5$, respectively.

The variations of the *u*-*x* transformation function with respect to *u* are shown in Fig. 1 for the results obtained using the exact transformation, the present accurate third-moment transformation (16), the Cornish-Fisher expansion (7), and the transformation function (14) obtained from HOMST. Fig. 1 reveals the following:

1. The transformation function, (7), obtained using the Cornish-Fisher expansion provides results that appear wavelike compared to the exact results. When the absolute value of *u* is large, the results obtained from (7) differ greatly from the exact results, indicating that, although the *u*-*x* transformation obtained from the Cornish-Fisher expansion is quite complicated, it does not provide appropriate *u*-*x* transformation results for this example.
2. Eq. (14) yields significant errors when *u* is less than -1 and cannot provide real results when *u* is larger than 2.8, for this example.
3. The proposed transformation function (16) generally provides better results than other functions, even though only the information of the first three moments is used, whereas other formulas use the information of the first four moment informations. However, when *u* is less than -2 for this example, the present formula also yields significant errors.

The variations of the transformed standard normal random variable *u* with respect to the standardized random variable x_s are shown in Fig. 2 for the results obtained using the exact transformation, the proposed accurate third-moment transformation function (21), the transformation function (6) obtained from the inverse Cornish-Fisher expansion, the transformation function (12) obtained from HOMST, and the transformation function (5) obtained from the Edgeworth expansion. Fig. 2 reveals the following:

1. Both the transformation function obtained from the Edgeworth expansion and that obtained from the inverse Cornish-Fisher expansion provide results that appear wavelike compared to the exact results. When *u* is large, the results obtained using the inverse Cornish-Fisher expansion differ greatly from the exact results. That is to say, obtaining appropriate *x*-*u* transformation results using the Edgeworth and Cornish-Fisher expansions is difficult, and the procedure itself is quite complicated.

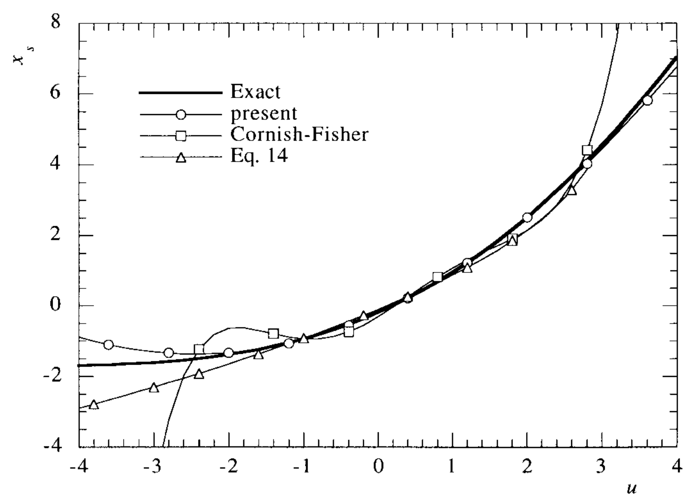

FIG. 1. ***u*-*x* Transformation for Gamma Random Variable**

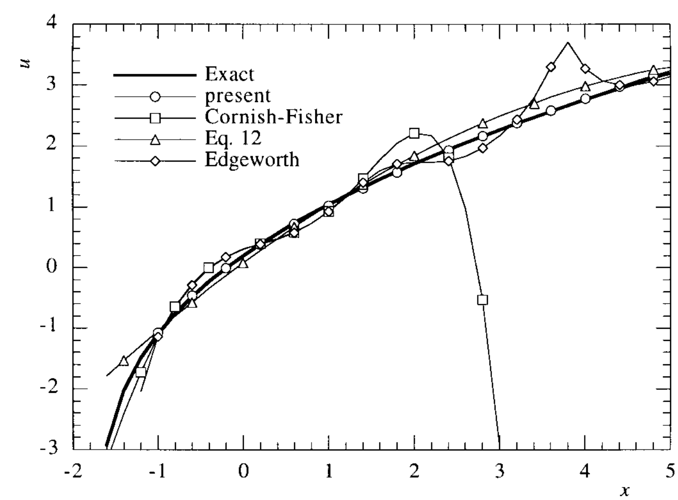

FIG. 2. ***x*-*u* Transformation for Gamma Random Variable**

2. Good agreement was obtained between the results obtained using the proposed transformation function (21) and the exact results, even though the proposed transformation function uses only the information of the first three moments, whereas other formulas use the information of the first four moments.

u-*x* and *x*-*u* Transformations for Function of Random Variables

Example 2 considers the following function of random variables:

$$X = R - S \tag{24}$$

where R is a normal random variable with $\mu_R = 3.0$ and $\sigma_R = 0.6$; and S is a Weibull random variable with $\mu_S = 2.0$ and $\sigma_S = 1.0$. Using the first four moments of R and S, the first two moments of X can be obtained as $\mu_x = 1.0$ and $\sigma_x = 1.166$ and the skewness and kurtosis of X can be readily obtained as $\alpha_3 = -0.357$ and $\alpha_4 = 3.071$. Because the PDF/CDf of X is unknown, the exact x-u transformation can be obtained using the following integration:

$$u = \Phi^{-1}\left[\int_0^{\infty} \Phi\left[\frac{t + x - \mu_R}{\sigma_R}\right] f_W(t)\, dt\right] \tag{25}$$

where f_W is the PDF of the Weibull distribution; and Φ is the CDF of the unit normal distribution.

The variations of the u-x and x-u transformation functions are shown in Figs. 3 and 4, respectively, for the results obtained using the exact transformation, the proposed accurate third-moment transformations (16) and (21), the Edgeworth and Cornish-Fisher expansions (5)–(7), and the transformation functions (12) and (14) obtained from HOMST. Figs. 3 and 4 show that all of the methods in the investigation provide good approximations of the exact results when the absolute value of u or x_s is small. When the absolute value of u or x_s is large, only the results provided by the proposed formula are relatively close to the exact results.

From the two aforementioned examples, one can see that although the Edgeworth and Cornish-Fisher expansions are quite complicated and require higher-order moments, the accuracy of their transformations is not good. This is because the absolute values of the skewness for the two examples are too large ($\alpha_3 = 1.1547$, -0.357 for examples 1 and 2, respectively) to satisfy the fundamental requirement for so-called ''mild nonnormality.'' This can also explain why the results of the Edgeworth and Cornish-Fisher expansions in example 2 are better than those in example 1.

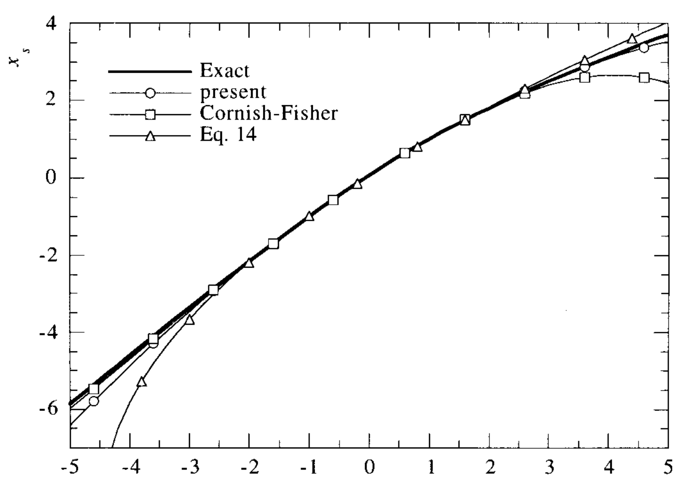

FIG. 3. ***u*-*x* Transformation for *X* = *R*-*S* where *R*, *S* Are Normal and Weibull Random Variables, Respectively**

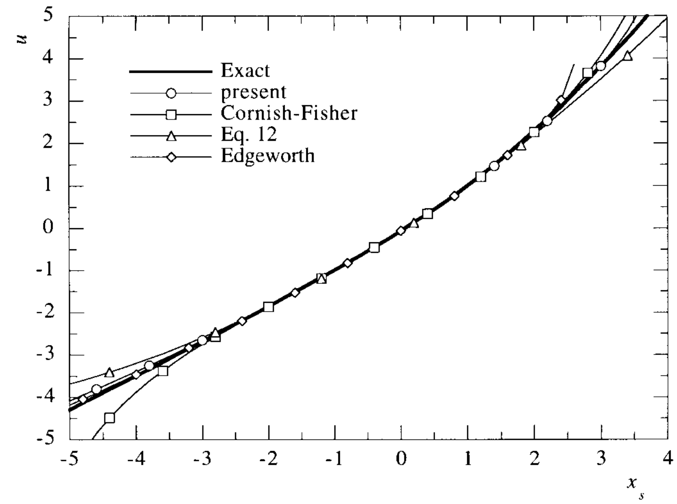

FIG. 4. ***x*-*u* Transformation for *X* = *R*-*S* where *R*, *S* Are Normal and Weibull Random Variables, Respectively**

Reliability Analysis Using FORM for dR-S Reliability Model

Example 3 considers the following performance function, which is an elementary reliability model that has several applications:

$$G(\mathbf{X}) = dR - S \tag{26}$$

where R = resistance having $\mu_R = 500$ and $\sigma_R = 100$; S = load with coefficient of variation of 0.4; and d = modification of R having normal distribution, $\mu_d = 1$ and $\sigma_d = 0.1$.

The following four cases are investigated under the assumption that R and S follow different probability distributions:

- Case 1: R is lognormal and S is Weibull (Type III—smallest)
- Case 2: R is gamma and S is lognormal
- Case 3: R is Weibull and S is lognormal
- Case 4: R is Frechet (Type II—largest) and S is exponential

Because all of the random variables in the performance function (26) have a known PDF/CDF, the first-order reliability index for the four cases described above can be readily obtained using FORM. In order to investigate the efficiency of the proposed reliability method, including random variables with unknown PDF/CDFs, the PDF/CDF of random variable R in the four cases is assumed to be unknown, and only its first three moments are known. Considering the first three moments, the u-x and x-u transformations in FORM can be performed easily using the proposed method, and then the first-order reliability index, including random variables that have an unknown PDF/CDF, can also be readily obtained.

The skewnesses of R corresponding to cases 1–4 are easily obtained as 0.608, 0.4, -0.352, and 2.353, respectively. The first-order reliability index obtained using the PDF/CDF of R and using only the first three moments of R are depicted in Fig. 5 for mean values of S in the range of 100–500. Fig. 5 reveals that, for all four cases, the results of the first-order reliability index obtained using only the first three moments of R are very close to those obtained using the PDF/CDF of R. This is to say that the proposed method is accurate enough to include random variables with an unknown PDF/CDF.

For case 3, the detailed results obtained while determining the design point using the PDF/CDF of R and the first three moments of R are listed in Table 2. Table 2 shows that the

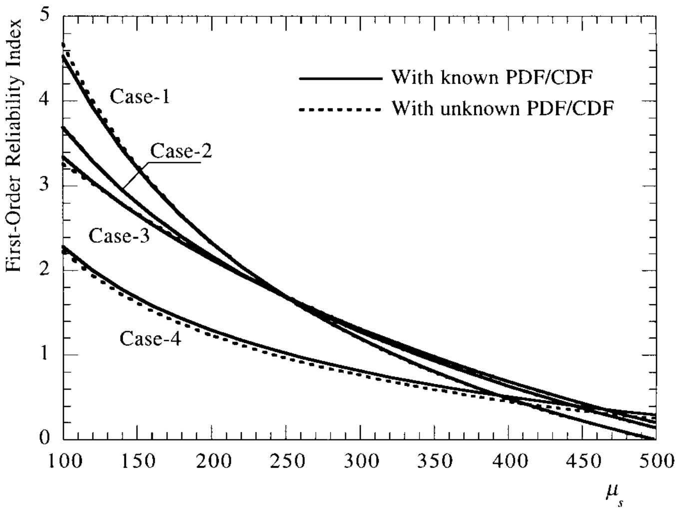

FIG. 5. Comparison of First-Order Reliability Index with Known and Unknown PDF/CDF for Example 3

checking point (in original and standard normal space), the Jacobians, and the first-order reliability index obtained in each iteration using the first three moments of R [columns (6)–(9)] are generally close to those obtained in each iteration using the PDF/CDF of R [columns (2)–(5)].

Application in Point-Fitting SORM

The fourth example considers the following performance function, a plastic collapse mechanism of a one-bay frame, which has been used by Der Kiureghian (1987).

$$G(\mathbf{X}) = x_1 + 2x_2 + 2x_3 + x_4 - 5x_5 - 5x_6 \quad (27)$$

The variables x_i are statistically independent and lognormally distributed and have means of $\mu_1 = \cdots = \mu_4 = 120$, $\mu_5 = 50$, and $\mu_6 = 40$, respectively, and standard deviations of $\sigma_1 = \cdots = \sigma_4 = 12$, $\sigma_5 = 15$, and $\sigma_6 = 12$, respectively.

Because all of the random variables in the performance function shown in (27) have a known PDF/CDF, the reliability index can be readily obtained using FORM/SORM. The FORM reliability index is $\beta_F = 2.348$, which corresponds to a failure probability of $P_F = 0.00943$. The true value of failure probability is $P_F = 0.0121$ (Der Kiureghian 1987). Using the point-fitting SORM, the point-fitted performance function is obtained as

$$G'(u) = 273.08 + 11.91u_1 + 23.81u_2 + 23.81u_3 + 11.91u_4 - 54.82u_5 - 51.00u_6 + 0.584u_1^2 + 1.147u_2^2 + 1.147u_3^2 + 0.584u_4^2 - 17.91u_5^2 - 12.08u_6^2 \quad (28)$$

The second-order reliability index is obtained as $\beta_S = 2.273$, which corresponds to a failure probability of $P_F = 0.0115$ (Zhao and Ono 1999a).

In order to investigate the application of the proposed reliability method, including random variables with unknown PDF/CDFs to the point-fitting SORM, the PDF/CDFs of random variable x_5 and x_6 are assumed to be unknown, and only their first three moments are known. Using the first three moments, the u-x and x-u transformations can be performed easily using the proposed method, and then the point-fitting SORM, including random variables with unknown PDF/CDF, can also be performed easily. The point-fitted performance function is obtained as

$$G'(u) = 287.63 + 11.91u_1 + 23.81u_2 + 23.81u_3 + 11.91u_4 - 73.13u_5 - 58.50u_6 + 0.583u_1^2 + 1.146u_2^2 + 1.146u_3^2 + 0.583u_4^2 - 11.78u_5^2 - 9.425u_6^2 \quad (29)$$

The design point in u-space, the average curvature radius, and the first- and second-reliability indices obtained using the first three moments of x_5 and x_6 are listed in Table 3, along with a comparison of those obtained using the PDF/CDFs of x_5 and x_6. Table 3 reveals that the results obtained using the first three moments of x_5 and x_6 are very close to those obtained using the PDF/CDFs of x_5 and x_6. This is to say that the proposed u-x and x-u transformations are applicable to the point-fitting SORM.

Shortcomings of Proposed Method

In order to investigate the shortcomings of the present method, the fifth example considers a Weibull random variable. Two cases, that is, when the coefficient of variation is taken to be $V = 0.1$ and $V = 0.7$, are investigated. As listed in Table 1, the skewness α_3 is equal to -0.715 for $V = 0.1$ and 1.131 for $V = 0.7$.

The variations of the u-x transformation function with respect to u are shown in Fig. 6 for the results obtained from the exact transformation, the proposed accurate third-moment transformation (16). Fig. 6 shows that the proposed method provides good approximations for the exact result when the absolute value of u is not very large. For $V = 0.1$, which implies that the skewness is negative, the proposed method produces significant error when u is larger than 2. For $V = 0.7$, which implies that the skewness is positive, the proposed method produces significant errors when u is less than -3, for this example. Needless to say, the specific range of u or x_s in which the method provides good approximations depends on the value of α_3. As shown in examples 3 and 4, appropriate results can be generally obtained using the method.

TABLE 2. Comparison of FORM Procedure with Known and Unknown PDF/CDF for Example 3

	USING PDF/CDF				USING FIRST THREE MOMENTS			
	Checking Point		Jacobian		Checking Point		Jacobian	
Iteration (1)	(x) (2)	(u) (3)	(dx/du) (4)	β (5)	(x) (6)	(u) (7)	(dx/du) (8)	β (9)
1	1 500 150	0 −0.0683 0.1926	0.1 101.71 57.788	0.2044	1 500 150	0 −0.0588 0.1926	0.1 100.35 57.788	0.2014
2	0.8863 271.67 214.82	−1.1371 −2.0865 1.1249	0.1 115.89 82.759	2.6291	0.8846 276.59 215.91	−1.1536 −2.0766 1.1381	0.1 124.07 83.182	2.6340
5	0.9508 293.42 278.98	−0.4922 −1.9002 1.8033	0.1 117.35 107.49	2.6655	0.9529 284.75 271.33	−0.4713 −1.9863 1.7311	0.1 123.01 104.53	2.6766

TABLE 3. Comparison of SORM Results for Example 4

Method (1)	Design point **U*** (2)	β_F (3)	R (4)	β_S (5)	P_f (6)
With known PDF/CDF	{−0.181, −0.356, −0.356, −0.181, 1.890, 1.274}	2.348	−33.17	2.273	0.0115
With unknown PDF/CDF	{−0.184, −0.362, −0.362, −0.184, 1.830, 1.313}	2.325	−49.18	2.274	0.0115

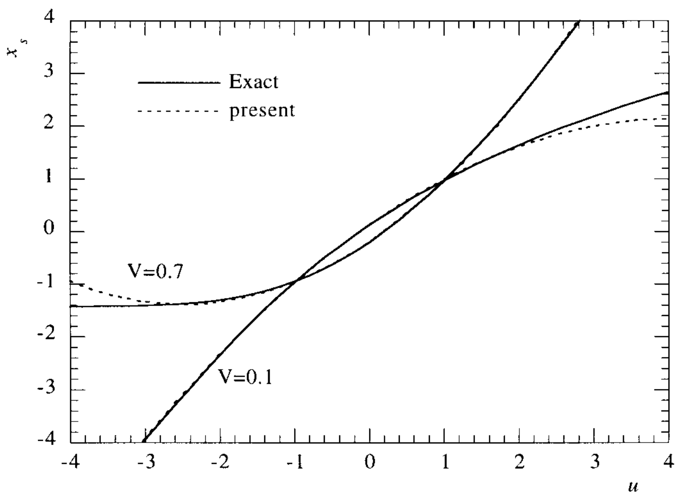

FIG. 6. *u-x* Transformation for Weibull Random Variable

CONCLUSIONS

1. The proposed method enables the u-x and x-u transformations for random variables with unknown PDF/CDF to be realized, thus eliminating the necessity of using the Rosenblatt transformation. From numerical examples, the proposed method is found to be accurate enough to include random variables with unknown PDF/CDF in reliability analysis using FORM/SORM.
2. Despite being very simple, the proposed third-moment standardization function provides more appropriate u-x and x-u transformation results compared to the other existing high-order moment transformation functions.
3. The range of the first moment ratio, that is, the skewness α_3, for which the proposed third-moment standardization is operable, is $-2\sqrt{2} \le \alpha_3 \le 2\sqrt{2}$. The limitation is not strict for general engineering use. Further study is required for problems that include random variables that have an extremely large absolute value of skewness.
4. The proposed method produces significant error for an extremely large u or x_s, when the skewness is negative and for an extremely small u or x_s when the skewness is positive. However, the proposed method generally provides more appropriate results in wider ranges of u or x_s than other existing methods.

APPENDIX I. EDGEWORTH AND CORNISH-FISHER EXPANSIONS

Let $\{F_n\}$ be a sequence of distribution functions depending on a parameter n, and converging to the normal distribution Φ as n increases, then for $\{L\} = \{l_r = k_r - \delta_r, r = 1, 2, \ldots\}$, where k_r is the rth cumulant, $\delta_r = 1$ if $r = 2$, otherwise $\delta_r = 0$, the following three expansions can be written (Stuart and Ord 1987)

$$F_n(x) \approx \Phi(x) - \phi(x) \sum_{r=1}^{\infty} M_r(x, L) \tag{30}$$

$$u = \Phi^{-1}[F_n(x)] \approx x - \sum_{r=1}^{\infty} N_r(x, L) \tag{31}$$

$$x = F_n^{-1}[\Phi(u)] \approx u + \sum_{r=1}^{\infty} P_r(u, L) \tag{32}$$

where Φ and ϕ are the distribution and density functions, respectively, of a standard normal random variable. Eqs. (30) and (32) are usually referred to as Edgeworth and Cornish-Fisher expansions, and (31) is referred to as an inverse Cornish-Fisher expansion (Stuart and Ord 1987).

Without loss of generality, using the standardized random variable in (4), l_1 and l_2 will become 0, (30), (31), and (32) can be rewritten in terms of the cumulants of x_s, $K = \{k_r, r > 2\}$, and the polynomials can be simplified greatly. Using the first four polynomials of the Edgeworth and Cornish-Fisher expansions in terms of $L = \{l_r, r > 0\}$ (Withers 1984), the first four explicit polynomials of $M_r(x, K)$ in the Edgeworth expansion (30) are obtained as

$$M_1(x, K) = \frac{1}{6} h_2 k_3 \tag{33a}$$

$$M_2(x, K) = \frac{1}{24} h_3 k_4 + \frac{1}{72} h_5 k_3^2 \tag{33b}$$

$$M_3(x, K) = \frac{1}{120} h_4 k_5 + \frac{1}{144} h_6 k_3 k_4 + \frac{1}{1{,}296} h_8 k_3^3 \tag{33c}$$

$$M_4(x, K) = \frac{1}{720} h_5 k_6 + \frac{1}{5{,}760} h_7 (5k_4^2 + 8k_3 k_5) + \frac{1}{1{,}728} h_9 k_3^2 k_4 + \frac{1}{31{,}104} h_{11} k_3^4 \tag{33d}$$

where h_j is the jth Hermite polynomial expressed as

$$h_j = (-1)^j \phi(x) \frac{d^j \phi(x)}{dx^j} \tag{34}$$

The first four explicit polynomials of $N_r(x, K)$ in the inverse Cornish-Fisher expansion (31) are expressed as

$$N_1(x, K) = \frac{1}{6} k_3 (x^2 - 1) \tag{35a}$$

$$N_2(x, K) = \frac{1}{24} k_4 (x^3 - 3x) - \frac{1}{36} k_3^2 (4x^3 - 7x) \tag{35b}$$

$$N_3(x, K) = \frac{k_5}{120} (x^4 - 6x^2 + 3) - \frac{k_3 k_4}{144} (11x^4 - 42x^2 + 15) + \frac{k_3^3}{648} (69x^4 - 187x^2 + 52) \tag{35c}$$

$$N_4(x, K) = \frac{k_6}{720} (x^5 - 10x^3 + 15x) - \frac{k_4^2}{384} (5x^5 - 32x^3 + 35x) - \frac{k_3 k_5}{360} (7x^5 - 48x^3 + 51x) + \frac{k_3^2 k_4}{864} (111x^5 - 547x^3 + 456x) - \frac{k_3^4}{7{,}776} (948x^5 - 3{,}628x^3 + 2{,}473x) \tag{35d}$$

The first four explicit polynomials of $P_r(x, K)$ in the Cornish-Fisher expansion (32) are expressed as

$$P_1(u, K) = \frac{k_3}{6} (u^2 - 1) \tag{36a}$$

$$P_2(u, K) = \frac{k_4}{24} (u^3 - 3u) - \frac{k_3^2}{36} (2u^3 - 5u) \tag{36b}$$

$$P_3(u, K) = \frac{k_5}{120}(u^4 - 6u^2 + 3) - \frac{k_3k_4}{24}(u^4 - 5u^2 + 2) + \frac{k_3^3}{324}(12u^4 - 53u^2 + 17) \quad (36c)$$

$$P_4(u, K) = \frac{k_6}{720}(u^5 - 10u^3 + 15u) - \frac{k_4^2}{384}(3u^5 - 24u^3 + 29u) - \frac{k_3k_5}{180}(2u^5 - 17u^3 + 21u) + \frac{k_3^2k_4}{288}(14u^5 - 103u^3 + 107u) - \frac{k_3^4}{7{,}776}(252u^5 - 1{,}688u^3 + 1{,}511u) \quad (36d)$$

If only the information of the first four moments of x_s and the relationships $\alpha_3 = k_3$ and $\alpha_4 = k_4 + 3$ are used, then (5)–(7) are obtained.

As indicated at the beginning of this section, the fundamental requirement of the Edgeworth and Cornish-Fisher expansions is mild nonnormality.

APPENDIX II: DERIVATION OF (17)

Making the first three moments of $S_u(u)$ equal to those of x_s, the following equations containing a_1, a_2, and a_3 are obtained:

$$\mu_{su} = a_1 + a_3 = 0 \quad (37a)$$

$$\sigma_{su}^2 = a_2^2 + 2a_3^2 = 1 \quad (37b)$$

$$\alpha_{3su}\sigma_{su}^2 = 6a_2^2a_3 + 8a_3^3 = \alpha_{3x} \quad (37c)$$

After simplification, the following reduced cubic equation of a_3 is obtained:

$$a_3^3 - \frac{3}{2}a_3 + \frac{1}{4}\alpha_3 = 0 \quad (38)$$

Eq. (38) has three real roots when $\alpha_3^2 \le 8$ and one real root when $\alpha_3^2 > 8$. From (37b), a_3 should satisfy the following equation in order for a_2 to be real:

$$a_3 \le \frac{\sqrt{2}}{2} \quad (39)$$

The only root of (38) that satisfies (39) is obtained as

$$a_3 = \text{sign}(\alpha_3)\sqrt{2}\cos\left[\frac{\text{sign}(\alpha_3)\theta' - \pi}{3}\right] \quad (40)$$

where

$$\theta' = \arctan\left(\frac{\sqrt{8 - \alpha_3^2}}{-\alpha_3}\right) \quad (41)$$

Using the relationships $\tan(-\varphi) = -\tan(\varphi)$ and $\cos(-\varphi) = \cos(\varphi)$, (17) can be readily obtained.

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APPENDIX IV. NOTATION

The following symbols are used in this paper:

a_1, a_2, a_3 = coefficients used in third-moment standardization function;
$f(\mathbf{X})$ = joint probability density function of $\mathbf{X}$;
$G(\mathbf{X})$ = performance function;
$h_j(x)$ = jth order Hermite polynomial;
k_r = rth cumulant;
M_r = rth polynomial of Edgeworth expansion;
N_r = rth polynomial of inverse Cornish-Fisher expansion;
P_f = probability of failure;

P_r = rth polynomial of Cornish-Fisher expansion;
R = resistance;
S = load;
$\mathbf{U}$ = standard normal random variables;
$\mathbf{X}$ = random variables;
x_s = random variable corresponding to x with its mean value and deviation standardized;
α_{3x} = first moment ratio, i.e., coefficient of skewness of random variable x;
α_{4x} = second moment ratio, i.e., coefficient of kurtosis of random variable x;
β_F = first-order reliability index;
β_S = second-moment reliability index;
μ = mean value;
σ = standard deviation;
$\Phi(x)$ = standard normal probability distribution with argument x; and
$\phi(x)$ = standard normal density distribution with argument.